Chapter 3 Derivatives and differentials

Before studying this chapter, we define the **change in a function** (or **function increment**). Let the variable x change from its initial value x_1 to its final value x_2. The difference between the final value and the initial value is $x_2 - x_1$. It is called the **change in the variable** x, denoted by $\Delta x = x_2 - x_1$.

For the function $y = f(x)$, when the independent variable x changes from x_1 to $x_2 = x_1 + \Delta x$, the corresponding change in the function y is Δy, as shown in Figure 3.1, this is expressed as

$$\Delta y = y_2 - y_1 = f(x_2) - f(x_1) = f(x_1 + \Delta x) - f(x_1)$$

Note: The change can be positive or negative.

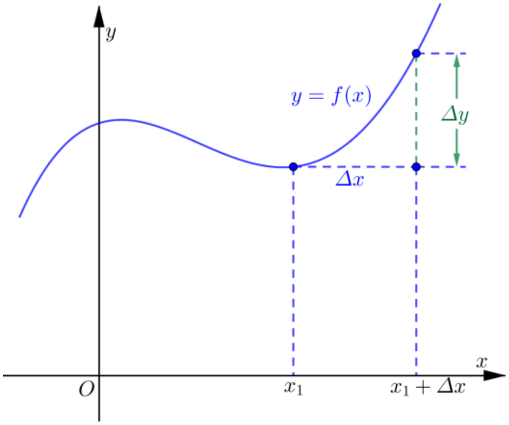


Figure 3.1

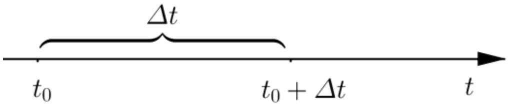
3.1. Two examples of the derivative concept

When solving real-world problems, we often need to understand more than just the functional relationship between variables; we also need to study the rate of change. Examples include the velocity of a moving object, the rate of population growth in a city, or the pace of a country's economic development. To address these problems, we introduce the concept of the derivative to better describe how these quantities change. Let us consider two practical examples.

(1) Instantaneous velocity of non-uniform linear motion

If s denotes the displacement travelled by an object in linear motion from an initial time to time t, then s is a function of time t, denoted by $s = f(t)$.

We now consider the velocity of the object at the specific time $t = t_0$, denoted by $v(t_0)$. When the time changes from t_0 to $t_0 + \Delta t$, as shown in Figure 3.2,



$\Delta t = (t_0 + \Delta t) - t_0$

Figure 3.2

the displacement that the object travels over the time interval Δt is

$$\Delta s = f(t_0 + \Delta t) - f(t_0)$$

If the object undergoes uniform motion, its velocity, $v(t)$, does not change with time. In this case

$$v(t_0) = \frac{\Delta s}{\Delta t} = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t} = \text{constant}$$

Here, $v(t_0)$ is a constant. It is the velocity of the object at time t_0, which is identical to the velocity of the object at any time t. That is, $v(t) = v(t_0) = \text{constant}$.

However, when the object moves with varying velocity, its velocity $v(t)$ changes over time. In this case, the ratio $\frac{\Delta s}{\Delta t}$ represents the average velocity, denoted by $\bar{v}$, over the time interval from t_0 to $t_0 + \Delta t$:

$$\bar{v} = \frac{\Delta s}{\Delta t} = \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

When Δt is very small, $\bar{v}$ can be used to approximate the velocity of the object at time t_0, denoted by $v(t_0)$. As Δt becomes smaller, the approximation becomes more accurate. As $\Delta t \to 0$, if the limit $\lim\limits_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$ exists, it is called the instantaneous velocity of the object at time t_0. That is,

$$v(t_0) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t_0)}{\Delta t}$$

(2) Tangent

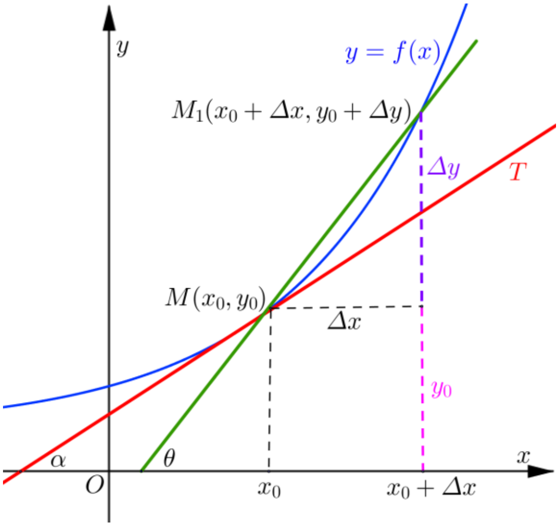


Figure 3.3

Let $M(x_0, y_0)$ be a fixed point on the curve $y = f(x)$. And let $M_1(x_0 + \Delta x, y_0 + \Delta y)$ be another point on the same curve. The position of M_1 is determined by Δx, making it a moving point. The line connecting M and M_1 is called a chord. Let its angle of inclination (the angle between the chord MM_1 and the x-axis) be θ.

From Figure 3.3, it can be seen that the gradient of the chord MM_1 is

$$\tan\theta = \frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

As $\Delta x \to 0$, the moving point M_1 approaches the fixed point M along the curve. Consequently, the chord MM_1 changes and approaches its limiting position, represented by the line MT. We define the line MT as the tangent to the curve at the point M. It follows that θ approaches the angle of inclination α of the tangent MT.

That is, the gradient of the tangent MT is

$$\tan\alpha = \lim_{\Delta x \to 0} \tan\theta = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

Although the two examples above have different practical meanings, speaking from the abstract numeric relationship, their mathematical nature is the same. They can both be generalised as the calculation of the ratio of the change in the function to the change in the independent variable, and the determination of the limit as the change in the independent variable approaches zero. The limit of this specific ratio $\frac{0}{0}$ is called the **derivative** of the function.

3.2. Concept of the derivative

Definition of the derivative

Let the function $y = f(x)$ be defined in a neighbourhood of x_0. When the independent variable x undergoes an increment Δx at the point x_0, the function $f(x)$ undergoes a corresponding increment.

$$\Delta y = f(x_0 + \Delta x) - f(x_0)$$

If the limit of $\frac{\Delta y}{\Delta x}$ exists as $\Delta x \to 0$, that is,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \qquad \text{exists}$$

then this limit is called the **derivative** or **differential** of the function $f(x)$ at x_0. This can be denoted by

$$f'(x_0) \qquad y'\big|_{x=x_0} \qquad \left.\frac{dy}{dx}\right|_{x=x_0} \qquad \left.\frac{df(x)}{dx}\right|_{x=x_0}$$

The ratio $\frac{\Delta y}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$ represents the average speed of change in the function $f(x)$ as the independent variable x changes from x_0 to $x_0 + \Delta x$, which is called the average rate of change in the function. The derivative $f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$ represents the instantaneous rate of change of the function at x_0. This is simply referred to as the rate of change of the function at x_0.

Example. Find the derivative of the function $y = x^2$ at $x = 2$.

Solution. When x changes from 2 to 2 + Δx, the increment in the function is

$$\Delta y = f(2 + \Delta x) - f(2) = (2 + \Delta x)^2 - 2^2 = 4\Delta x + (\Delta x)^2$$

Therefore,

$$\frac{\Delta y}{\Delta x} = 4 + \Delta x$$

$$f'(2) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (4 + \Delta x) = 4 \qquad \square$$

If the function $f(x)$ has a derivative at x_0, we say that the function $f(x)$ is **differentiable** at x_0 or that the derivative $f'(x_0)$ exists. Otherwise, we say that the function $f(x)$ is not differentiable at x_0 or that the derivative $f'(x_0)$ does not exist. If $f'(x_0) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \pm\infty$, in this case, it cannot be treated that $f'(x_0)$ exists because $\pm\infty \notin R$, hence, the derivative $f'(x_0)$ does not exist. However, for convenience, we sometimes say that the derivative of function at x_0, $f'(x_0)$, is infinity. If the function $f(x)$ is differentiable at every point within the interval (a, b), then we say that the function $f(x)$ is differentiable on (a, b).

If $f(x)$ is differentiable on (a, b), then at each point x in the interval (a, b), there is a corresponding derivative value. Consequently, a new function can be defined, which is called the derivative function of the function $y = f(x)$ with respect to x on (a, b). This is often referred to simply as the derivative, and is denoted by

$$y' \qquad f'(x) \qquad \frac{dy}{dx} \qquad \frac{df(x)}{dx}$$

That is, $$y' = f'(x) = \frac{dy}{dx} = \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

According to the definition of the derivative, the two examples in Section 3.1 can be stated as follows:

(1) Instantaneous velocity is the derivative of displacement s with respect to time t, that is,

$$v(t) = s' = \frac{ds}{dt}$$

(2) The gradient of the tangent MT to the curve $y = f(x)$ at x is the derivative of the ordinate (the y-coordinate) of the curve with respect to the abscissa (the x-coordinate).

$$\tan\alpha = f'(x) = \frac{dy}{dx}$$

3.3. Differentials

Above, we use $\frac{dy}{dx}$ to denote the derivative, that is,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

This equality is a proportion.

Proof. Let $f(x)$ be a differentiable function of x and its derivative is

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

From $\Delta y = \frac{\Delta y}{\Delta x} \cdot \Delta x$ we can obtain

$$\lim_{\Delta x \to 0} \Delta y = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \Delta x = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \cdot \lim_{\Delta x \to 0} \Delta x = f'(x) \cdot 0 = 0$$

That is, $f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x} = \frac{0}{0}$

Hence, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x} = \frac{0}{0}$

That is, $\frac{dy}{dx} = \frac{\lim_{\Delta x \to 0} \Delta y}{\lim_{\Delta x \to 0} \Delta x}$ is a proportion.

The derivative $f'(x)$ is a limit of the $\frac{0}{0}$ type.

Definition of the differential

Because $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x)$, if $\Delta x \to 0$, then we have

$$\frac{\Delta y}{\Delta x} \to f'(x) \qquad \text{or} \qquad [\frac{\Delta y}{\Delta x} - f'(x)] \to 0$$

Let $\frac{\Delta y}{\Delta x} - f'(x) = \mathrm{a}$ $\qquad \mathrm{a} \to 0$

Then $\Delta y = f'(x)\, \Delta x + \mathrm{a}\Delta x$

That is, $\Delta y - \mathrm{a}\Delta x = f'(x)\, \Delta x$

Let $dy = \Delta y - \mathrm{a}\Delta x,$ $\qquad dx = \Delta x$

Then $dy = f'(x)\, dx,$ $\qquad \frac{dy}{dx} = f'(x)$

Since $dx = \Delta x \to 0$

thus dx is an infinitesimal, $0 < |dx| \le 0.05$.

The value of dy is determined by the value of $f'(x)\,dx$.

Generally, both dy and dx are infinitesimals.

dy is called the differential of the function $f(x)$, and dx is called the differential of the independent variable x.

Having introduced the concept of the differential, we can see that $\frac{dy}{dx}$ represents the quotient of the differential of the function dy and the differential of the independent variable dx. Consequently, the derivative $f'(x)$ is also known as the quotient of differentials. Since problems of finding the differential $dy = f'(x)\,dx$ is essentially the problems of finding derivatives, methods for finding derivatives and differentials are called differentiation methods.

Example 1. Find the differential dy of the function $y = x^2$ when x changes from 1 to 1.01.

Solution. Because $\frac{dy}{dx} = \frac{dx^2}{dx} = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0}(2x + \Delta x) = 2x$

it follows that $dy = 2xdx$

From the given condition

$$dx = \Delta x = 1.01 - 1 = 0.01$$

Here, $dx = \Delta x$ signifies that the differential dx of the independent variable x is equal to its increment Δx. Therefore, the differential of the function is

$$dy = 2xdx = 2 \cdot 1 \cdot (0.01) = 0.02$$

Geometric significance of the differential and its application

Sketch the graph of the function $y = f(x)$ on the xy–plane, as shown in Figure 3.4.

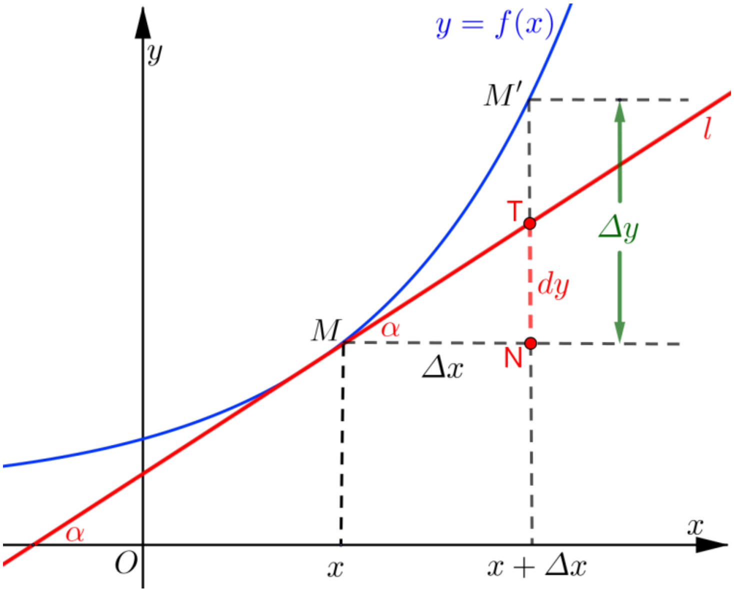


Figure 3.4

Let $M(x, y)$ be a fixed point on the curve. Draw the tangent l to the curve at the point M. The gradient of this tangent l is

$$\tan\alpha = f'(x)$$

When the independent variable x undergoes an increment Δx, we obtain another point on the curve, $M'(x + \Delta x, y + \Delta y)$. From Figure 3.4, it can be seen that

$$\mathrm{N}M = \Delta x \qquad \mathrm{N}M' = \Delta y \qquad \tan\alpha = \frac{\mathrm{TN}}{\mathrm{N}M}$$

That is, $$\mathrm{TN} = \mathrm{N}M \cdot \tan\alpha = f'(x) \cdot \Delta x = dy \qquad (\Delta x = dx)$$

Therefore, the differential dy of the function $y = f(x)$ represents the increment in the ordinate (the y-coordinate) of the tangent l at the point $M(x, y)$. In the graph, the segment $\mathrm{T}M'$ represents the difference between Δy and dy.

That is, $$\mathrm{T}M' = \mathrm{e} = \Delta y - dy, \qquad \text{where} \qquad \mathrm{e} = \mathrm{a}\Delta x$$

When $|\Delta x|$ is very small, e is also very small and can be neglected.

That is, $$\Delta y - dy = \mathrm{e} \approx 0$$

Therefore, the formula for calculating an approximation using differentials is

$$\Delta y \approx dy = f'(x)\,\Delta x \qquad (\Delta x = dx)$$

Since $\Delta y = f(x+\Delta x) - f(x)$

it can be written as

$$f(x+\Delta x) - f(x) \approx f'(x)\,\Delta x$$

Alternatively, $f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x$

Example 2. Find an approximation for the value of $\sqrt[3]{1.02}$.

Solution. This problem can be approached by finding an approximation for the value of the function $f(x) = \sqrt[3]{x}$ at $x = 1.02$.

Since $f'(x) = \dfrac{1}{3\sqrt[3]{x^2}}$ (Differentiation process is omitted.)

thus $f(x+\Delta x) \approx f(x) + f'(x)\,\Delta x = \sqrt[3]{x} + \dfrac{1}{3\sqrt[3]{x^2}} \cdot \Delta x$

Let $x = 1$ and $\Delta x = 0.02$ (This satisfies the requirement that Δx is very small.)

Therefore, $f(1+0.02) = f(1.02) = \sqrt[3]{1.02} \approx \sqrt[3]{1} + \dfrac{1}{3\sqrt[3]{1^2}} \cdot 0.02 \approx 1.0067$

That is, $\sqrt[3]{1.02} \approx 1.0067$

Example 3. A spherical ball has an outer diameter of 10 cm. The thickness of the spherical shell is $\dfrac{1}{20}$ cm. Find an approximation for the value of the volume of the spherical shell.

Solution. The radius of the ball is $r = \dfrac{10}{2} = 5$.

The thickness of the shell of the ball is $dr = \Delta r = -\frac{1}{20}$.

The volume of the ball is

$$V = f(r) = \frac{4}{3}\pi r^3$$

The derivative is $f'(r) = 4\pi r^2$ (Differentiation process is omitted.)

The volume of the spherical shell is ΔV, then

$$\Delta V \approx dV = f'(r)dr = 4\pi r^2 \cdot dr = 4\pi \cdot 5^2 \cdot (-\frac{1}{20}) = -5\pi$$

That is, the required volume of the spherical shell is

$$|\Delta V| = 5\pi \text{ cm}^3$$

3.4. Fundamental derivative formulae and computation rules

From the definition of the derivative, the steps for finding a derivative can be summarised as follows:

Step 1. Determine the increment in the function, Δy, corresponding to the increment Δx in the independent variable x.

$$\Delta y = f(x+\Delta x) - f(x)$$

Step 2. Form the ratio of the increments.

$$\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Step 3. Calculate the limit of $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$, that is,

$$y' = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

In the definition of derivative, we have already clearly explained the essence of the concept of derivative. The method above for finding the derivative $f'(x)$ of the function $f(x)$ is based directly on the definition (i.e. differentiating from first principles). However, if, for every function, we have to find its derivative directly according to the definition, it can be cumbersome and complex. We therefore seek to find some fundamental formulae and computation rules to simplify the computation in finding derivatives.

a) Derivative of a constant

Function $y = \mathrm{c}$ (c is a constant.)

$$\Delta y = f(x+\Delta x) - f(x) = \mathrm{c} - \mathrm{c} = 0$$

Since $\Delta y = 0$ for any value of Δx, it follows that the ratio $\frac{\Delta y}{\Delta x} = 0$.

Therefore, $y' = \mathrm{c}' = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x \to 0} 0 = 0$

That is, $\mathrm{c}' = 0$

The derivative of a constant is equal to 0.

b) Derivatives of power functions

Function $y = x^n$ (n is a positive integer.)

From the binomial theorem, we know that

$$\begin{aligned}\Delta y &= (x + \Delta x)^n - x^n \\ &= [x^n + nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n] - x^n \\ &= nx^{n-1}\Delta x + \frac{n(n-1)}{2}x^{n-2}(\Delta x)^2 + \ldots + (\Delta x)^n\end{aligned}$$

Therefore, $\frac{\Delta y}{\Delta x} = nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}\Delta x + \ldots + (\Delta x)^{n-1}$

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = nx^{n-1}$$

That is, $(x^n)' = nx^{n-1}$

Later in this chapter, it will be shown that when n is any real number a, this formula remains valid. That is, for the function

$$y = x^a \qquad y' = ax^{a-1} \qquad a \in \mathrm{R}$$

c) Derivatives of sums and differences

If $u = u(x)$ and $v = v(x)$ are differentiable functions of x, then their sum (or difference) $y = u \pm v$ is also a differentiable function of x. Moreover,

$$y' = (u \pm v)' = u' \pm v'$$

Proof. Consider y as a function of two independent variables, u and v.

That is, $y = f(u, v) = u \pm v$

Therefore, $\Delta y = f(u + \Delta u, v + \Delta v) - f(u, v)$

$$= [(u + \Delta u) \pm (v + \Delta v)] - (u \pm v) = \Delta u \pm \Delta v$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} \pm \frac{\Delta v}{\Delta x}$$

Hence, $y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \pm \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta x} = u' \pm v'$

That is, $(u \pm v)' = u' \pm v'$

The formula can be applied to the sums and differences of a finite number of functions. That is,

$$(u_1 + u_2 + \ldots + u_n)' = u_1' + u_2' + \ldots + u_n'$$

Example 1. Find the derivative of the function $y = x^3 - x^2$.

Solution. $y' = (x^3 - x^2)' = (x^3)' - (x^2)' = 3x^{3-1} - 2x^{2-1} = 3x^2 - 2x$

d) Derivative of a product

If both $u = u(x)$ and $v = v(x)$ are differentiable functions of x, then their product $y = uv$ is also a differentiable function of x. Moreover,

$$y' = (uv)' = uv' + u'v$$

Proof. Consider y as a function of two independent variables, u and v, that is,

$$y = f(u, v) = uv$$

Hence, $\Delta y = f(u + \Delta u, v + \Delta v) - f(u, v)$

$$= (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\,\Delta u + \Delta u\Delta v$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \frac{\Delta u}{\Delta x}\Delta v$$

As $\Delta x \to 0$, values of u and v do not change, as u and v depend on x, rather than Δx. Furthermore, since v is differentiable, it is necessarily continuous. It follows that

$$\lim_{\Delta x \to 0} \Delta v = 0.$$

Therefore, $y' = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = u \lim\limits_{\Delta x \to 0} \frac{\Delta v}{\Delta x} + v \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} + \lim\limits_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \lim\limits_{\Delta x \to 0} \Delta v$

$$= uv' + vu' + u' \cdot 0$$

$$= uv' + u'v$$

That is, $(uv)' = uv' + u'v$

Especially when $u = \mathrm{c}$ (c is a constant)

$$y' = (\mathrm{c}v)' = \mathrm{c}v'$$

That is, a constant factor can be moved outside of the derivative operator.

This formula can be generalised to the product of a finite number of functions. That is, if $y = u_1 u_2 \cdots u_n$, then

$$(u_1 u_2 \cdots u_n)' = u_1' u_2 \cdots u_n + u_1 u_2' \cdots u_n + \ldots + u_1 u_2 \cdots u_{n-1} u_n'$$

Example 2. Find the derivative of the function $y = (1 + 2x)(3x^3 - 2x^2)$.

Solution.

$$\begin{aligned} y' &= (1 + 2x)(3x^3 - 2x^2)' + (1 + 2x)'(3x^3 - 2x^2) \\ &= (1 + 2x)[(3x^3)' - (2x^2)'] + [1' + (2x)'] (3x^3 - 2x^2) \\ &= (1 + 2x)[3(x^3)' - 2(x^2)'] + [0 + 2(x)'] (3x^3 - 2x^2) \\ &= (1 + 2x)(9x^2 - 4x) + 2(3x^3 - 2x^2) \\ &= 24x^3 - 3x^2 - 4x \end{aligned}$$

e) Derivative of a quotient

If u and v are differentiable functions of x and $v \neq 0$, then the quotient $y = \dfrac{u}{v}$ is also a differentiable function of x. Moreover,

$$y' = \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

Proof. Consider y as a function of two independent variables, u and v. That is,

$$y = f(u, v) = \frac{u}{v}$$

Hence,

$$\begin{aligned} \Delta y &= f(u + \Delta u, v + \Delta v) - f(u, v) \\ &= \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} = \frac{\Delta u \cdot v - u \cdot \Delta v}{v(v + \Delta v)} \end{aligned}$$

$$\frac{\Delta y}{\Delta x} = \frac{\frac{\Delta u}{\Delta x}\cdot v - u\cdot\frac{\Delta v}{\Delta x}}{v(v+\Delta v)}$$

Since as $\Delta x \to 0$, values of u and v do not change while $\Delta v \to 0$.

Therefore, $$y' = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \frac{\lim\limits_{\Delta x\to 0}\frac{\Delta u}{\Delta x}\cdot v - u\cdot\lim\limits_{\Delta x\to 0}\frac{\Delta v}{\Delta x}}{v(v+\lim\limits_{\Delta x\to 0}\Delta v)} = \frac{u'v-uv'}{v^2}$$

That is, $$(\frac{u}{v})' = \frac{u'v-uv'}{v^2}$$

Especially when $u = c$ (c is a constant)

$$(\frac{c}{v})' = c(\frac{1}{v})' = c\cdot\frac{1'v-1\cdot v'}{v^2} = -c\frac{v'}{v^2}$$

This formula demonstrates that the formula $y' = (x^n)' = nx^{n-1}$ is also valid when n is a negative integer.

In fact, when $n = \text{-}m$ ($m > 0$), that is, n is a negative integer,

$$y = x^n = x^{-m} = \frac{1}{x^m}$$

therefore, $$y' = (x^{-m})' = (\frac{1}{x^m})' = -\frac{(x^m)'}{(x^m)^2} = -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1}$$

Example 3. Find the derivative of the function $y = \frac{x^4}{3} - \frac{4}{x^3}$.

Solution. $$y' = (\frac{x^4}{3} - \frac{4}{x^3})' = (\frac{x^4}{3})' - (\frac{4}{x^3})' = \frac{1}{3}(x^4)' - 4(x^{-3})'$$

$$= \frac{4}{3}x^3 + 12x^{-4} = \frac{4}{3}x^3 + \frac{12}{x^4}$$

Example 4. Find the derivative of the function $y = \frac{x^2 - 1}{x^2 + 1}$.

Solution.

$$y' = (\frac{x^2-1}{x^2+1})' = \frac{(x^2-1)'(x^2+1)-(x^2-1)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(2x)(x^2+1)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

f) Derivatives of logarithm functions

Function $y = \log_a x \qquad (a > 0,\ a \neq 1)$

From $\Delta y = \log_a(x+\Delta x) - \log_a x = \log_a(1+\frac{\Delta x}{x})$

we obtain $\frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\log_a(1+\frac{\Delta x}{x}) = \frac{1}{x}\cdot\frac{x}{\Delta x}\log_a(1+\frac{\Delta x}{x}) = \frac{1}{x}\log_a(1+\frac{\Delta x}{x})^{\frac{x}{\Delta x}}$

Let $b = \frac{\Delta x}{x}$. Then as $\Delta x \to 0,\ b \to 0$.

Thus $\lim\limits_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim\limits_{\Delta x\to 0}\frac{1}{x}\log_a(1+\frac{\Delta x}{x})^{\frac{x}{\Delta x}} = \frac{1}{x}\log_a\lim\limits_{b\to 0}(1+b)^{\frac{1}{b}}$

$$= \frac{1}{x}\log_a e = \frac{1}{x\ln a} \qquad (\log_a b = \frac{\log_c b}{\log_c a})$$

where $\lim\limits_{b\to 0}(1+b)^{\frac{1}{b}} = e$ is an important limit.

Therefore, $y' = (\log_a x)' = \frac{1}{x}\log_a e = \frac{1}{x\ln a}$

Especially when $a = e$

$$y' = (\log_a x)' = (\ln x)' = \frac{1}{x}\log_e e = \frac{1}{x}\ln e = \frac{1}{x}, \quad \text{where} \quad \ln e = 1$$

That is, $y' = (\ln x)' = \frac{1}{x}$

This is the frequently used derivative formula for natural logarithms.

g) Derivatives of trigonometric functions

(1) Derivative of $y = \sin x$

From $\Delta y = \sin(x + \Delta x) - \sin x = 2\cos(x + \frac{\Delta x}{2})\sin\frac{\Delta x}{2}$

we obtain $$\frac{\Delta y}{\Delta x} = 2\cos(x + \frac{\Delta x}{2})\frac{\sin\frac{\Delta x}{2}}{\Delta x} = \cos(x + \frac{\Delta x}{2})\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

From the continuity of cos$x$, we have

$$\lim_{\Delta x \to 0} \cos(x + \frac{\Delta x}{2}) = \cos x$$

From the limit formula $\lim_{x \to 0}\frac{\sin x}{x} = 1$, we have

$$\lim_{\Delta x \to 0} \frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = 1$$

Hence, $$y' = \lim_{\Delta x \to 0} \cos(x + \frac{\Delta x}{2})\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = \lim_{\Delta x \to 0} \cos(x + \frac{\Delta x}{2})\lim_{\Delta x \to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = \cos x$$

That is, $(\sin x)' = \cos x$

Similarly, we can show that

(2) Derivative of $y = \cos x$

$$(\cos x)' = -\sin x$$

(3) Derivative of $y = \tan x$

$$y' = (\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x}$$

$$= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

That is, $(\tan x)' = \sec^2 x$

It is not difficult to derive the following:

(4) Derivatives of $y = \cot x$, $y = \sec x$, $y = \csc x$

$$(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$(\sec x)' = \left(\frac{1}{\cos x}\right)' = \sec x \cdot \tan x$$

$$(\csc x)' = \left(\frac{1}{\sin x}\right)' = -\csc x \cdot \cot x$$

Example 5. Find the derivative of the function $y = 2\sqrt{x} \cdot \sin x + \cos x \cdot \ln x$.

Solution. $y' = (2\sqrt{x} \cdot \sin x + \cos x \cdot \ln x)'$

$$= (2\sqrt{x} \cdot \sin x)' + (\cos x \cdot \ln x)'$$

$$= 2(\sqrt{x})'\sin x + 2\sqrt{x}(\sin x)' + (\cos x)'\ln x + \cos x(\ln x)'$$

$$= 2\cdot\frac{1}{2\sqrt{x}}\sin x + 2\sqrt{x}\cos x - \sin x\ln x + \frac{1}{x}\cos x$$

$$= \left(\frac{1}{\sqrt{x}} - \ln x\right)\sin x + \left(2\sqrt{x} + \frac{1}{x}\right)\cos x$$

h) Derivatives of composite functions (The Chain Rule)

To find a derivative easily, we usually treat a function $f(x)$ as a composite function,

that is, $f(x) \quad \Leftrightarrow \quad f(u), \quad u = u(x)$

For example,

$$y = \sqrt{x+1} = \sqrt{u}, \qquad u = x + 1$$

On the other hand, by the multiplication rule, we have

$$a = a \cdot 1 = a \cdot 1 \cdot 1 \cdot 1 \cdot \ldots$$

Hence, we get the derivative formula for composite functions, that is, the Chain Rule.

$$y' = \frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{du}{du} = \frac{dy}{dx}\cdot\frac{du}{du}\cdot\frac{dv}{dv}\cdot\ldots$$

$$= \frac{dy}{du}\cdot\frac{du}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}\cdot\ldots$$

where $\quad \frac{du}{du} = 1, \quad \frac{du}{du}\cdot\frac{dv}{dv}\cdot\ldots = 1\cdot 1\cdot\ldots = 1.$

Example 6. Find the derivative of $y = (1 + 2x)^{30}$.

Solution. **Method 1** (write down the substitution)

Let $\quad u = 1 + 2x \quad$ then $\quad y = (1 + 2x)^{30} = u^{30}$

Therefore, $\quad y' = \frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{du}{du} = \frac{dy}{du}\cdot\frac{du}{dx} = 30u^{30-1}\cdot 2$

where $\quad \frac{dy}{du} = 30u^{30-1}, \quad \frac{du}{dx} = 2.$

Hence, $\quad y' = 60(1 + 2x)^{29}$

Method 2 (direct method without writing down the substitution)

$$y' = \frac{dy}{dx} = \frac{d(1+2x)^{30}}{dx} = \frac{d(1+2x)^{30}}{dx}\cdot\frac{d(1+2x)}{d(1+2x)} = \frac{d(1+2x)^{30}}{d(1+2x)}\cdot\frac{d(1+2x)}{dx}$$

$$= 30(1 + 2x)^{30-1}\cdot 2 = 60(1 + 2x)^{29}$$

Example 7. Find the derivative of the function $y = \ln(\sin x)$.

Solution. $y' = \dfrac{dy}{dx} = \dfrac{d\ln(\sin x)}{dx} \cdot \dfrac{d\sin x}{d\sin x} = \dfrac{d\ln(\sin x)}{d\sin x} \cdot \dfrac{d\sin x}{dx} = \dfrac{1}{\sin x} \cdot \cos x = \cot x$

Example 8. Find the derivative of the function $y = \cos(nx)$.

Solution. $y' = \dfrac{dy}{dx} = \dfrac{d\cos(nx)}{dx} \cdot \dfrac{d(nx)}{d(nx)} = \dfrac{d\cos(nx)}{d(nx)} \cdot \dfrac{d(nx)}{dx} = -n\sin(nx)$

Example 9. Find the derivative of the function $y = (\dfrac{x}{2x+1})^n$.

Solution.

$$y' = \frac{dy}{dx} = \frac{d(\frac{x}{2x+1})^n}{dx} \cdot \frac{d\frac{x}{2x+1}}{d\frac{x}{2x+1}} = \frac{d(\frac{x}{2x+1})^n}{d\frac{x}{2x+1}} \cdot \frac{d\frac{x}{2x+1}}{dx}$$

$$= n(\frac{x}{2x+1})^{n-1} \cdot \frac{x'(2x+1) - x(2x+1)'}{(2x+1)^2}$$

$$= n(\frac{x}{2x+1})^{n-1} \cdot \frac{2x+1-2x}{(2x+1)^2} = \frac{nx^{n-1}}{(2x+1)^{n+1}}$$

Example 10. Find the derivative of the function $y = \sin(x^2+4)$.

Solution. $y' = \dfrac{dy}{dx} = \dfrac{d\sin(x^2+4)}{d(x^2+4)} \cdot \dfrac{d(x^2+4)}{dx} = \cos(x^2+4) \cdot 2x$

Example 11. Find the derivative of the function $y = \left[\ln(2x)\right]^3$.

Solution. $$y'=\frac{dy}{dx}=\frac{d\left[\ln(2x)\right]^3}{d\ln(2x)}\cdot\frac{d\ln(2x)}{d2x}\cdot\frac{d2x}{dx}$$

$$=3\left[\ln(2x)\right]^2\cdot\frac{1}{2x}\cdot 2=\frac{3\left[\ln(2x)\right]^2}{x}$$

Example 12. Find the derivative of the function $y=\cos\left[\ln(3x^2+6)^5\right]$.

Solution. $$y'=\frac{dy}{dx}=\frac{d\cos\left[\ln(3x^2+6)^5\right]}{d\ln(3x^2+6)^5}\cdot\frac{d\ln(3x^2+6)^5}{d(3x^2+6)^5}\cdot\frac{d(3x^2+6)^5}{d(3x^2+6)}\cdot\frac{d(3x^2+6)}{dx}$$

$$=-\sin\left[\ln(3x^2+6)^5\right]\cdot\frac{1}{(3x^2+6)^5}\cdot 5(3x^2+6)^4\cdot 6x$$

$$=-\frac{10x\sin\left[\ln(3x^2+6)^5\right]}{x^2+2}$$

Example 13. Find the derivative of the function $y=\sqrt{\cos\sqrt{x}}$.

Solution. $$y'=\frac{dy}{dx}=\frac{d\left(\cos\sqrt{x}\right)^{\frac{1}{2}}}{d\cos\sqrt{x}}\cdot\frac{d\cos\sqrt{x}}{d\sqrt{x}}\cdot\frac{d\sqrt{x}}{dx}$$

$$=\frac{1}{2}\left(\cos\sqrt{x}\right)^{-\frac{1}{2}}\cdot(-\sin\sqrt{x})\cdot\frac{1}{2}x^{-\frac{1}{2}}=-\frac{\sin\sqrt{x}}{4\sqrt{x\cos\sqrt{x}}}$$

Example 14. Find the derivative of the function $y=\sin\left[x^2\cos(3x)\right]$.

Solution. $$y'=\frac{dy}{dx}=\frac{d\sin\left[x^2\cos(3x)\right]}{dx^2\cos(3x)}\cdot\frac{dx^2\cos(3x)}{dx}$$

$$=\cos\left[x^2\cos(3x)\right]\cdot\left[(x^2)'\cos(3x)+x^2\cdot\frac{d\cos(3x)}{d3x}\cdot\frac{d3x}{dx}\right]$$

$$= \cos\left[x^2 \cos(3x)\right] \cdot \left[2x\cos(3x) - 3x^2 \sin(3x)\right] \qquad \square$$

By applying the Chain Rule for composite functions, we can also obtain the derivative formulae of a product and a quotient.

$$y = uv \qquad u = u(x) \qquad v = v(x)$$

Take logarithms of both sides.

$$\ln y = \ln(uv) = \ln u + \ln v$$

Differentiate both sides of the equation with respect to x.

$$\frac{d\ln y}{dx} = \frac{d\ln u}{dx} + \frac{d\ln v}{dx}$$

$$\frac{d\ln y}{dy} \cdot \frac{dy}{dx} = \frac{d\ln u}{du} \cdot \frac{du}{dx} + \frac{d\ln v}{dv} \cdot \frac{dv}{dx}$$

Hence, $$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx}$$

Substitute $y = uv$ and rearrange.

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Therefore, the derivative formula for a product is

$$y' = (uv)' = vu' + uv'$$

Similarly, we can also derive the derivative formula for a quotient, which is omitted here.

Generally speaking, apart from elementary functions, for functions $f(x)$ of all other forms, we can always use the Chain Rule for composite functions to find their derivatives.

i) Derivatives of implicit functions

Suppose the equation $\mathrm{P}(x, y) = 0$ defines y as a function of x, where the function y

is differentiable. The method of finding derivatives of composite functions (the Chain Rule) can also be applied to find derivatives of implicit functions y with respect to x.

Example 15. Given the equation $x^2 + y^2 - r^2 = 0$, where y is a function of x. Find the derivative y'.

Solution. Differentiate both sides with respect to x.

$$\frac{d(x^2)}{dx} + \frac{d(y^2)}{dx} - \frac{d(r^2)}{dx} = \frac{d0}{dx}$$

$$2x + \frac{d(y^2)}{dy} \cdot \frac{dy}{dx} - 0 = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

Divide both sides by 2.

$$x + y \cdot \frac{dy}{dx} = 0$$

$$y \cdot \frac{dy}{dx} = -x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

That is, $$y' = \frac{dy}{dx} = -\frac{x}{y}$$

Example 16. Given the equation $y^2 = 2px$, where y is a function of x. Find the derivative y'.

Solution. Differentiate both sides with respect to x.

$$\frac{d(y^2)}{dx} = \frac{d(2px)}{dx}$$

$$\frac{d(y^2)}{dy}\cdot\frac{dy}{dx}=2p$$

$$2y\cdot\frac{dy}{dx}=2p$$

$$\frac{dy}{dx}=\frac{p}{y}$$

Therefore, $$y'=\frac{dy}{dx}=\frac{p}{y}$$

Example 17. Given the equation $y=x\ln y$, where y is a function of x. Find the derivative y'.

Solution. Differentiate both sides with respect to x.

$$\begin{aligned} y' &= (x)'\ln y + x(\ln y)' \\ &= \ln y + x\cdot\frac{d(\ln y)}{dx} \\ &= \ln y + x\cdot\frac{d(\ln y)}{dy}\cdot\frac{dy}{dx} \\ &= \ln y + x\cdot\frac{1}{y}\cdot y' \end{aligned}$$

Rearrange. $$y'-\frac{x}{y}y'=\ln y$$

$$y'\left(\frac{y-x}{y}\right)=\ln y$$

Hence, $$y'=\frac{y\ln y}{y-x}$$

Example 18. Given the equation $x^2y^2+6xy+y^2=3$, where y is a function of x. Find the derivative y'.

Solution. Differentiate both sides with respect to *x*.

$$(x^2y^2)'+(6xy)'+(y^2)'=3'$$

$$\frac{d\left(x^2y^2\right)}{dx}+\frac{d\left(6xy\right)}{dx}+\frac{d(y^2)}{dx}=\frac{d3}{dx}$$

$$\frac{dx^2}{dx}\cdot y^2+x^2\cdot\frac{dy^2}{dy}\cdot\frac{dy}{dx}+6\cdot\frac{dx}{dx}\cdot y+6x\cdot\frac{dy}{dx}+\frac{dy^2}{dy}\cdot\frac{dy}{dx}=0$$

$$2xy^2+2x^2y\frac{dy}{dx}+6y+6x\frac{dy}{dx}+2y\frac{dy}{dx}=0$$

$$(2x^2y+6x+2y)\frac{dy}{dx}=-2xy^2-6y$$

$$y'=\frac{dy}{dx}=\frac{-2xy^2-6y}{2x^2y+6x+2y}=\frac{-xy^2-3y}{x^2y+3x+y}$$

j) Derivatives of inverse functions

If the function $y=f(x)$ has a non-zero derivative $f'(x)$ at the point x, and its inverse function $x=f^{-1}(y)$ is continuous at the corresponding point, then the derivative $\left[f^{-1}(y)\right]'$ exists and is given by

$$\left[f^{-1}(y)\right]'=\frac{1}{f'(x)} \quad\text{or}\quad f'(x)=\frac{1}{[f^{-1}(y)]'} \quad\text{or}\quad f'(x)\cdot\left[f^{-1}(y)\right]'=1$$

That is, $\dfrac{dx}{dy}=\dfrac{1}{\dfrac{dy}{dx}}$ or $\dfrac{dy}{dx}=\dfrac{1}{\dfrac{dx}{dy}}$ or $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$

Proof. Since $\dfrac{dy}{dy}\cdot\dfrac{dx}{dx}=1\cdot1=1$

thus $\dfrac{dy}{dx}\cdot\dfrac{dx}{dy}=1$ that is, $f'(x)\cdot\left[f^{-1}(y)\right]'=1$

k) Derivatives of inverse trigonometric functions

(1) Derivative of function $y = \arcsin x \quad (-1 < x < 1)$

Since the inverse function of $y = \arcsin x \quad (-1 < x < 1)$ is

$$x = \sin y \qquad (-\frac{\pi}{2} < y < \frac{\pi}{2})$$

thus

$$\frac{dx}{dy} = \frac{d\sin y}{dy} = \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$y' = (\arcsin x)' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{\sqrt{1-x^2}}$$

That is,

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

Similarly, it can be shown that

(2)

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}} \qquad (-1 < x < 1)$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(\text{arccot}\, x)' = -\frac{1}{1+x^2}$$

Example 19. Find the derivative of the function $y = \arcsin(3x^2)$.

Solution.

$$y' = \frac{dy}{dx} = \frac{d\arcsin(3x^2)}{dx}\cdot\frac{d(3x^2)}{d(3x^2)} = \frac{d\arcsin(3x^2)}{d(3x^2)}\cdot\frac{d(3x^2)}{dx}$$

$$= \frac{1}{\sqrt{1-(3x^2)^2}}\cdot 6x = \frac{6x}{\sqrt{1-9x^4}}$$

Example 20. Find the derivative of the function $y = \arctan\frac{1}{x}$.

Solution. $y' = \dfrac{dy}{dx} = \dfrac{d\arctan\frac{1}{x}}{dx} \cdot \dfrac{d\frac{1}{x}}{d\frac{1}{x}} = \dfrac{d\arctan\frac{1}{x}}{d\frac{1}{x}} \cdot \dfrac{d\frac{1}{x}}{dx} = \dfrac{1}{1+(\frac{1}{x})^2} \cdot (-\dfrac{1}{x^2}) = -\dfrac{1}{1+x^2}$

l) Derivatives of exponential functions

Exponential function $y = a^x \quad (a > 0,\ a \neq 1)$

Take logarithms of both sides.

$$\ln y = x\ln a$$

Differentiate both sides with respect to x.

$$\frac{1}{y} y' = \ln a$$

That is, $y' = y\ln a = a^x\ln a$

Therefore, $y' = (a^x)' = a^x\ln a$

Especially when $a = e$, we have

$$(e^x)' = e^x\ln e = e^x \cdot 1 = e^x$$

That is, $(e^x)' = e^x$ This is a frequently used derivative formula.

Example 21. Find the derivative of the function $y = a^{-x}$.

Solution. $y' = \dfrac{dy}{dx} = \dfrac{d(a^{-x})}{dx} \cdot \dfrac{d(-x)}{d(-x)} = \dfrac{d(a^{-x})}{d(-x)} \cdot \dfrac{d(-x)}{dx} = a^{-x}\ln a \cdot (-1) = -a^{-x}\ln a$

Example 22. Find the derivative of the function $y = e^{ax^2+bx+c}$.

Solution. **Method 1** Take logarithms of both sides.

$$\ln y = (ax^2 + bx + c)\ln e$$

$$\ln y = ax^2 + bx + c$$

Differentiate both sides with respect to x.

$$\frac{1}{y}y' = 2ax + b$$

$$y' = (2ax + b)y$$

Therefore, $$y' = (2ax + b)\,e^{ax^2+bx+c}$$

Method 2

$$y' = \frac{de^{ax^2+bx+c}}{d(ax^2+bx+c)} \cdot \frac{d(ax^2+bx+c)}{dx} = e^{ax^2+bx+c} \cdot (2ax + b) = (2ax + b)\,e^{ax^2+bx+c}$$

Example 23. Given the equation $e^y = xy$, where y is a function of x. Find the derivative y'.

Solution. Differentiate both sides with respect to x.

$$\frac{de^y}{dx} = x'y + xy'$$

$$e^y y' = y + xy'$$

Rearranging gives $$y' = \frac{y}{e^y - x}$$

m) Differentiate with the method of taking logarithms (Logarithmic Differentiation)

In deriving the derivative formula for the exponential function above, we first take logarithms of both sides of $y = a^x$. We then differentiate both sides with respect to x to find the derivative y'. This technique is called differentiating with the method of taking logarithms (or logarithmic differentiation). It is a particularly useful method in finding the derivatives of some functions. The following examples illustrate this method further.

Example 24. Find the derivative of the function $y = x^x$.

Solution. This function is neither a power function nor an exponential function; it is a composite exponential function (where both the base and the exponent are variables). Consequently, we cannot directly apply the standard derivative formulae for power or exponential functions. Instead, we differentiate this by using the method of logarithmic differentiation.

Take logarithms of both sides.

$$\ln y = x \ln x$$

Differentiate both sides with respect to x.

$$\frac{1}{y}y' = x'\ln x + x(\ln x)'$$

$$= \ln x + x\cdot\frac{1}{x}$$

$$= \ln x + 1$$

Therefore, $$y' = y(\ln x + 1) = x^x(\ln x + 1)$$

Example 25. Find the derivative of the function $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)}}$.

Solution. Differentiating this directly using the Chain Rule would be unnecessarily complex. Consequently, we first take logarithms of both sides.

$$\ln y = \frac{1}{2}[\ln(x-1) + \ln(x-2) - \ln(x-3) - \ln(x-4)]$$

Differentiate both sides with respect to x.

$$\frac{1}{y}y' = \frac{1}{2}\left(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4}\right)$$

Therefore, $y' = \frac{1}{2} y(\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4})$

Example 26. Find the derivative of the power function $y = x^a \quad (a \in \mathrm{R})$.

Solution. Take logarithms of both sides.

$$\ln y = a \ln x$$

Differentiate both sides with respect to x.

$$\frac{1}{y} y' = a \cdot \frac{1}{x}$$

Hence, $y' = a \cdot \frac{y}{x} = a \cdot \frac{x^a}{x} = ax^{a-1}$

Therefore, $y' = (x^a)' = ax^{a-1}$

This is the frequently used derivative formula for power functions.

• Derivative formulae

For the ease of reference and use, the standard derivative formulae are listed below:

(1) $(c)' = 0$ (c is a constant.)

(2) $(u \pm v)' = u' \pm v'$

(3) $(uv)' = u'v + uv'$

(4) $(cu)' = cu'$ (c is a constant.)

(5) $(\frac{u}{v})' = \frac{u'v - uv'}{v^2}$ $(v \neq 0)$

(6) $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u)u'(x)$ where $y = f(u),\ u = u(x)$

(7) $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$ or $\left[f^{-1}(y)\right]' = \frac{1}{f'(x)}$ $(f'(x) \neq 0)$

(8) $(x^a)' = ax^{a-1}$

(9) $(\log_a x)' = \frac{1}{x} \log_a e$ $(a > 0,\ a \neq 1)$

(10) $(\ln x)' = \frac{1}{x}$

(11) $(a^x)' = a^x \ln a$ $\quad (a > 0)$

(12) $(e^x)' = e^x$

(13) $(\sin x)' = \cos x$

(14) $(\cos x)' = -\sin x$

(15) $(\tan x)' = \frac{1}{\cos^2 x} = \sec^2 x$

(16) $(\cot x)' = -\frac{1}{\sin^2 x} = -\csc^2 x$

(17) $(\sec x)' = \sec x \cdot \tan x$

(18) $(\csc x)' = -\csc x \cdot \cot x$

(19) $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$ $\quad (-1 < x < 1)$

(20) $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$ $\quad (-1 < x < 1)$

(21) $(\arctan x)' = \frac{1}{1+x^2}$

(22) $(\text{arccot}\, x)' = -\frac{1}{1+x^2}$

- Differential formulae

From $dy = f'(x)\,dx$, we can see that to find the differential dy, we simply calculate the derivative $f'(x)$ and multiply it by dx.

Based on the differentiation rules for the sum, product, and quotient of functions, we can derive the corresponding differential formulae for the sum, product and quotient of functions. For example, using the product rule for the function $y = uv$

$$y' = uv' + u'v$$

and multiplying both sides by dx, we obtain the differential formula:

$$dy = d(uv) = udv + vdu$$

The differential formulae are summarised as follows:

(1) $dc = 0$ (c is a constant.)

(2) $d(u \pm v) = du \pm dv$

(3) $d(uv) = udv + vdu$

(4) $d(cu) = cdu$ (c is a constant.)

(5) $d(\frac{u}{v}) = \frac{vdu - udv}{v^2}$ $(v \neq 0)$

(6) $d(x^a) = ax^{a-1}dx$

(7) $d(\log_a x) = \frac{1}{x}\log_a e dx$ $(a > 0, a \neq 1)$

(8) $d(\ln x) = \frac{1}{x} dx$

(9) $d(a^x) = a^x \ln a dx$ $(a > 0)$

(10) $d(e^x) = e^x dx$

(11) $d(\sin x) = \cos x dx$

(12) $d(\cos x) = -\sin x dx$

(13) $d(\tan x) = \frac{1}{\cos^2 x} dx = \sec^2 x dx$

(14) $d(\cot x) = -\frac{1}{\sin^2 x} dx = -\csc^2 x dx$

(15) $d(\sec x) = \sec x \cdot \tan x dx$

(16) $d(\csc x) = -\csc x \cdot \cot x dx$

(17) $d(\arcsin x) = \frac{1}{\sqrt{1-x^2}} dx$ $(-1 < x < 1)$

(18) $d(\arccos x) = -\frac{1}{\sqrt{1-x^2}} dx$ $(-1 < x < 1)$

(19) $d(\arctan x) = \frac{1}{1+x^2} dx$

(20) $d(\text{arccot} x) = -\frac{1}{1+x^2} dx$

3.5. Higher derivatives

At the beginning of this chapter, we discussed the instantaneous velocity of an object moving in a straight line with varying speed. If the displacement of the object is given by the equation $s = f(t)$, then the instantaneous velocity of the object at time t is the derivative of s with respect to t, that is,

$$v = s' = f'(t)$$

Velocity $v = f'(t)$ is also a function of time t. Its derivative with respect to time t is called the instantaneous acceleration a at time t.

$$a = v' = (s')' = s''$$

This is known as the second derivative (or the second order derivative) of s with respect to t.

For example, the motion equation for an object in free fall is

$$s = \frac{1}{2}\mathrm{g}t^2$$

Hence, the instantaneous velocity is

$$v = s' = (\frac{1}{2}\mathrm{g}t^2)' = \mathrm{g}t$$

The instantaneous acceleration is

$$a = s'' = (\mathrm{g}t)' = \mathrm{g}$$

In general, if the derivative $f'(x)$ of the function $f(x)$ is differentiable at the point x, then the derivative of $f'(x)$ at the point x is called the second derivative (or second order derivative) of the function $f(x)$ at the point x, denoted by

$$f''(x) \qquad y'' \qquad \text{or} \qquad \frac{d^2y}{dx^2}$$

Similarly, the derivative of the second derivative $y'' = f''(x)$ is called the third derivative (or the third order derivative) of the function $y = f(x)$, denoted by

$$f'''(x) \qquad y''' \qquad \text{or} \qquad \frac{d^3y}{dx^3}$$

In general, we define the nth derivative of the function $y = f(x)$ as the derivative of the $(n-1)$th derivative of the function $y = f(x)$. That is,

$$\left[y^{(n-1)}\right]' = y^{(n)} \qquad (n = 2, 3, 4, \ldots)$$

denoted by $$f^{(n)}(x) \qquad y^{(n)} \qquad \text{or} \qquad \frac{d^n y}{dx^n}$$

All derivatives of order two and above are collectively known as higher derivatives (or higher order derivatives). The values of derivatives of all orders for the function $f(x)$, evaluated at the point $x = x_0$, are denoted by

$$f'(x_0) \qquad f''(x_0) \qquad \ldots \qquad f^{(n)}(x_0)$$

alternatively, $$y'\big|_{x=x_0} \qquad y''\big|_{x=x_0} \qquad \ldots \qquad y^{(n)}\Big|_{x=x_0}$$

Example 1. Find derivatives of all orders for the function $y = x^4$.

Solution. $$y' = 4x^3 \qquad y'' = 12x^2 \qquad y''' = 24x \qquad y^{(4)} = 24$$

$$y^{(5)} = y^{(6)} = \ldots = 0$$

Example 2. Find the nth derivative for the function $y = e^x$.

Solution. Since $(e^x)' = e^x$, the function does not change after differentiation, it follows that $y^{(n)} = e^x$

Example 3. Find the nth derivative for the function $y = \sin x$.

Solution. $y' = (\sin x)' = \cos x = \sin(x + \frac{\pi}{2})$

$$y'' = [\sin(x + \frac{\pi}{2})]' = \cos(x + \frac{\pi}{2}) = \sin(x + 2\cdot\frac{\pi}{2})$$

$$y''' = [\sin(x + 2\cdot\frac{\pi}{2})]' = \cos(x + 2\cdot\frac{\pi}{2}) = \sin(x + 3\cdot\frac{\pi}{2})$$

…

Generally, we have

$$y^{(n)} = (\sin x)^{(n)} = \sin(x + n\cdot\frac{\pi}{2})$$

Similarly, we can prove that

$$(\cos x)^{(n)} = \cos(x + n\cdot\frac{\pi}{2})$$

Exercise 3

1. Use the definition of the derivative to find the following derivatives.

a. $f(x) = x^2$ **b**. $f(x) = x^3 + x$

c. $f(x) = 1 - 2x^2$ **d**. $f(x) = \frac{1}{x}$

e. $f(x) = \sqrt{x}$ **f**. $f(x) = \frac{1}{x^2}$

2. Find the derivatives of the following functions.

a. $y = 2x^3 - x + 6$ **b**. $y = \frac{3}{x^3} + \frac{x^3}{3}$

c. $y = 5\sqrt{x} + \frac{2}{x} - \sqrt[5]{x^3}$ **d**. $y = \frac{x-1}{x+1}$

e. $y=(2x-3)\sqrt{x}$

f. $y=\dfrac{\sin x}{x^2}$

g. $y=\cos x\ln x$

h. $y=\tan(5x)+\sin(3x)-\sin^2 x$

i. $y=\sqrt{\cos(5x)}$

j. $y=\dfrac{5\ln x}{1+\cos x}$

k. $y=\dfrac{x}{\sin x+\cos x}$

l. $y=\sec(2x-3)\ln(3x+2)$

3. Find the derivatives of the following functions.

a. $y=\sin(3x)+\sin^3(2x)$

b. $y=\sin(x^3)+\log_2(x^2+1)$

c. $y=\cos(7x-5)-\sin(2x+3)$

d. $y=(2-x)(2+x^3)^2$

e. $y=(1-\sin x)^2$

f. $y=\tan^3(\frac{x}{3})+\sin(\cos\sqrt{x})$

g. $y=\sqrt{\dfrac{1+x^2}{1-x}}$

h. $y=\tan(5x)+\left[\ln x^2\right]^3$

i. $y=\sqrt{\cos(3x+5)}$

j. $y=\sec^5(2x+1)$

k. $y=\ln\sqrt{3x+4}+\sqrt{\ln(3x+4)}$

l. $y=(\ln x)^5+\dfrac{1}{\cos^4(2x)}$

4. Find the derivatives of the following functions.

a. $y=\tan\dfrac{5x}{4}\cos^3\dfrac{x}{2}$

b. $y=\sqrt[3]{\ln\dfrac{2x}{5}}$

c. $y=\ln(x\cos x)$

d. $y=(9+x^4)\sqrt{5-x}$

e. $y=(2+3x^2)\sqrt{1+4x^3}$

f. $y=\log_3(1+x^3)$

g. $y=\ln\dfrac{1-\sqrt{x}}{1+\sqrt{x}}$

h. $y=\tan^3(\log_3(4x+5))$

i. $y=\ln^3(\tan(2x+3))$

j. $y=\cos(2x^3\ln x)$

k. $y=\frac{\cos x}{\sqrt{1-x^2}}$

l. $y=x^6\sqrt[5]{x^7-2}$

5. Find the derivatives of the following functions.

a. $y=\sqrt[3]{\frac{1}{1-x^2}}$

b. $y=\frac{1}{\tan^2(3x)}$

c. $y=\sin(x^2\cos(3x))$

d. $y=\frac{\ln(\cos(2x))}{\sin^2 x}$

e. $y=\sin^3[\ln x\cos(5x)]$

f. $y=\frac{\ln^5 x}{\sec(5x)}$

g. $y=(x^4-x^3+2x)^7$

h. $y=(\frac{x-3}{2x+1})^4$

i. $y=(x^5+3x^2)^{\frac{3}{4}}$

j. $y=\sqrt{\cos ec(5x)}$

k. $y=\sec(\frac{3x}{1+x^2})$

l. $y=\ln[\sin^2(5x+2)]$

6. Find $\frac{dy}{dx}$ of the following.

a. $x^3+y^3=5$

b. $2xy-x^2y=9$

c. $x^3(2x-3y)=4$

d. $e^{xy}+y\ln x-\cos(3x)=1$

e. $\sin(x^2+y^2)=3x$

f. $y=1+xe^y$

g. $3xy^3+x^2y-y^2=7$

h. $\sqrt{x+y^2}=x$

i. $\ln(x^2+y)=x^4y-\cos x$

j. $e^y+8xy+x^2=7$

k. $x^y=y^x$

l. $\frac{1}{x^2}+\frac{1}{y}=-8$

7. Find the derivatives of the following functions.

a. $y=\frac{\sin x}{e^{5x}}$

b. $y=3^{\sqrt{x}}\arctan x^2$

c. $y = 5^{\sqrt{\cos(2x)}}$

d. $y = \dfrac{\arcsin(2x)}{\sin x + \cos x}$

e. $y = \ln(xe^{3x})$

f. $y = (\arccos \dfrac{x}{3})^4$

g. $y = e^{4x+3} \sin(e^{x^2})$

h. $y = \cos(2x^3 e^{-x})$

i. $y = \dfrac{\arccos x}{\sqrt{1-x^2}}$

j. $y = e^{\sin^2(3x)} \cdot \cos(e^{-2x})$

k. $y = \arctan^2(\dfrac{1}{2x})$

l. $y = \sin^3[e^{2x} \cos(5x)]$

m. $y = \dfrac{\ln^5 x}{e^{2x+3}}$

n. $y = \sin(2^{\sin(2x)})$

o. $y = e^{2-\cos^3(3x+5)}$

p. $y = \ln(2^{\sqrt{x}} - e^{-2x})$

q. $y = \arctan(\sqrt{x^5 - x})$

r. $y = \arccos \dfrac{x-1}{x+2}$

s. $y = e^{\arcsin \sqrt{2x}}$

t. $y = \cos \dfrac{\arcsin x}{5}$

u. $y = 3^{4^{x^3}}$

v. $y = 10^{3+x^4}$

w. $y = \sqrt[3]{\arctan \sqrt[3]{x}}$

z. $y = \arcsin(\cos^2 x)$

8. Find the derivatives of the following functions.

a. $y = x^{\cos x}$

b. $y = x\sqrt{\dfrac{1+x}{1-x}}$

c. $y = (\ln x)^x$

d. $y = \sqrt[3]{\dfrac{x^2+3}{x+2}}$

e. $y = (\tan x)^{\sin x}$

f. $y = (x^3+2)^{\frac{1}{4}} \cdot \sin^5 x \cdot 3^{x+2}$

g. $y = x^{x^{x^3}}$

h. $y = (\sin x)^{\cos^2 x}$

i. $y = \dfrac{\sqrt{4x+3}}{5^{x^3} \cos(2x)}$

j. $y = (2x+3)^5 \cdot (4x-1)^7 \cdot (x^3+2)^8$

9. Find the first and second derivatives of the following.

a. $y = \ln(1+x^3)$

b. $y = x^4 + 5x + 8$

c. $y = xe^{-2x}$

d. $y\sin x - \cos(x-y) = 0$

e. $e^x + \sin(x^2) = y$

f. $y^3 + 2\ln y = x^3$

g. $x + \arctan y = y$

h. $x^2 + y^2 = e^{\arctan y}$

10. Given that $xy - \sin(\pi y^2) = 0$, find the values for $y'\big|_{x=0,y=1}$ and $y''\big|_{x=0,y=1}$.

11. Find the differential, *dy*, for each of the following functions.

a. $y = 2x^4$

b. $y = \dfrac{x}{1-x^2}$

c. $y = e^{-x}\cos x$

d. $y = e^{\sin x}$

e. $xy = 5$

f. $y = 1 + xe^y$

12. Find an approximation for each of the following using differentials.

a. $\sqrt[5]{0.95}$

b. $\sqrt[3]{8.02}$

c. $\ln(1.01)$

d. $e^{0.05}$

13. Find the *n*th derivative of each of the following functions.

a. $y = 3^x$

b. $y = \ln(1+x)$

c. $y = e^{5x}$

d. $y = (2+x)^m$ where *m* is a constant.