University Calculus:

A New Perspective

For Any University and College Program

Volume 1

Spring Seeds

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Preface

The first chapter of this book gives preparatory knowledge on which calculus depends. The second chapter explains the concept of limit and its definition from a dynamic point of view. With this basic knowledge, further study of calculus becomes concrete and straightforward.

We have written this calculus textbook to use in our teaching, and it has received high praise from students. Students found the concepts and theories in this book to be very clear and logical, and thus, learning calculus became very easy, which is easier than learning other mathematics courses. Students also achieved excellent grades in the study of calculus. The teaching effect is remarkable. It can be said that this book has completely changed the difficult situation of teaching calculus and made learning calculus easy.

Calculus is one of the outstanding achievements of the human intellect. It is the foundation of modern mathematics. All advanced mathematics evolved from it and algebra, and consequently, it forms the basis of modern science, especially physics. It is a powerful tool for solving complex problems in mathematics, physics, economics, the social sciences, engineering, and the biological sciences, among other fields. The importance of calculus is obvious. Therefore, we have written this textbook in the hope that all calculus students will gain a true understanding and mastery of the subject and that it will provide strong support for further development in related fields in the future.

This textbook comprises 16 chapters and is divided into two volumes. To make this textbook fully adaptable, it contains essential contents applicable to different university and college programs, as well as contents marked with * that can be selected by various universities and colleges.

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Chapter 2 Limits

2.1. Limits of functions and variables

The concept and theory of limits are mathematical theories from a dynamic point of view. They form the foundation of calculus. The two fundamental concepts in calculus, differentiation and integration, are based on the concept of the limit.

The approach symbol $x \to x_0$ indicates the change process of the independent variable x approaching the limit value x_0.

from the left of x_0 $x \to x_0$ Or $x_0 \leftarrow x$ from the right of x_0

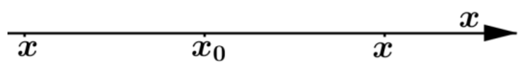

Figure 2.1

As shown in Figure 2.1, at the beginning, x takes value in the direction of x_0. In the end, x takes a value that is infinitely close to x_0. That is, at the end of the change process,

$$x = x_0 + a \qquad \text{or} \qquad x \approx x_0$$

where a is an infinitesimal.

An infinitesimal a is a variable that approaches zero, that is, $a \to 0$. At the end of the change process

$$a \approx 0 \qquad \Leftrightarrow \qquad 0 < |a| \leq 0.05$$

Using the above notion of the dynamic approach, we now provide an equally dynamic definition for the limit of a function:

If a function $f(x)$ is defined in a neighbourhood of x_0, when x approaches infinitely close to x_0, the corresponding functional value $f(x)$ will approach infinitely close to a certain constant A, then A is called the limit of the function $f(x)$ as $x \to x_0$, denoted by

$$\lim_{x \to x_0} f(x) = A$$

If $f(x)$ is a polynomial function or a fractional function whose denominator's limit is not zero as

$x \to x_0$, then we have

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The limit symbol, $\lim_{x \to x_0}(\bullet)$, means that the independent variable x in the expression $(\bullet)$ will finally take the value of $x = x_0$ in the change process when x approaches x_0.

For the change process of x as $x \to x_0$, the corresponding change process of $f(x)$ is $f(x) \to A$, this is generally denoted by

$$f(x) \to A \qquad (x \to x_0)$$

In summary

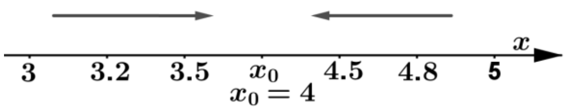

Figure 2.2

The approach symbol $x \to x_0$ and the limit symbol $\lim_{x \to x_0}(\bullet)$ mean that at the beginning, x takes value in the direction of x_0, either from the left of x_0 ($x < x_0$) or from the right of x_0 ($x > x_0$), as shown in Figure 2.2.

$$x = 3 \ldots 3.2 \ldots 3.5 \ldots \; x_0 - a, \; x_0$$

or

$$x = 5 \ldots 4.8 \ldots 4.5 \ldots \; x_0 + a, \; x_0$$

At the end of the change process

$$x = x_0 \pm a \approx x_0 \qquad (\text{for } x \to x_0)$$

$$x = x_0 \qquad (\text{for } \lim_{x \to x_0}(\bullet))$$

With the above concept of limits, we can now apply the limit methods to prove the indeterminate form $\frac{0}{0} = a$.

Proof. Let $\lim_{x \to 0} x = 0$

then $a \cdot \lim_{x \to 0} x = a \cdot 0 = 0$ $\qquad$ a is a random number.

Dividing both sides by $\lim_{x \to 0} x$ gives

$$a \cdot \frac{\lim\limits_{x\to 0} x}{\lim\limits_{x\to 0} x} = \frac{0}{\lim\limits_{x\to 0} x}$$

That is, $$a \cdot 1 = \frac{0}{0}$$

Hence, $$a = \frac{0}{0} = 0 \cdot \infty = \frac{\infty}{\infty} = \infty - \infty \qquad \square$$

Note: Above, we have given the concepts and definitions of approach and limit from a dynamic point of view. They should be easy to understand and grasp. However, in order to have a more comprehensive understanding of the nature of the limit, we will provide the definition of the limit from a static point of view below:

If, for any given positive number $\varepsilon > 0$, there always exists a positive number $\delta > 0$ such that whenever $0 < |x - x_0| < \delta$, the inequality

$$|f(x) - A| < \varepsilon$$

always holds, then the function $f(x)$ is said to have a limit at constant A when x approaches x_0. This is denoted by

$$\lim_{x\to x_0} f(x) = A \qquad \text{or} \qquad f(x) \to A \qquad (x \to x_0)$$

Here, the limit and approach have the same definition. However, their properties and calculation results are different. Care must be taken to distinguish the differences and do not confuse them. Furthermore, this definition only conforms to mathematical logic but lacks operability and is difficult to understand. $\square$

2.2. Computation rules of limits

As $x \to x_0$, the limits of the functions $f(x)$ and $g(x)$ are A and B, respectively.

That is, $$\lim_{x\to x_0} f(x) = A \qquad \lim_{x\to x_0} g(x) = B$$

Then (1) $$\lim_{x\to x_0} \left[f(x) \pm g(x)\right] = \lim_{x\to x_0} f(x) \pm \lim_{x\to x_0} g(x) = A \pm B$$

(2) $$\lim_{x\to x_0} \left[f(x) \times g(x)\right] = \lim_{x\to x_0} f(x) \times \lim_{x\to x_0} g(x) = A \times B$$

If $g(x) = b$ is a constant, then

$$\lim_{x\to x_0} bf(x) = b\lim_{x\to x_0} f(x) = bA$$

(3) $$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)} = \frac{A}{B} \qquad B\neq 0$$

- In the calculations above, the variable x is a continuous variable. For a discontinuous integer variable n, except $n\to\infty, \lim\limits_{n\to\infty}(\bullet)$, there is no other computation of limit.

Example 1. Find $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$.

Solution. As $x\to 2, x\neq 2$, the common factor $(x-2)$ can be cancelled from the numerator and the denominator.

Thus $$\lim_{x\to 2}\frac{x^2-4}{x-2} = \lim_{x\to 2}\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

(At the end of the change process, $x = 2$.)

Example 2. Find $\lim\limits_{x\to 3}\frac{x^2-9}{x^3-27}$.

Solution. Because $\lim\limits_{x\to 3}(x^3-27)=0$, the computation rule (3) cannot be used directly.

However, as $x\to 3, x\neq 3$. Both the numerator and denominator have a factor $(x-3)$. Factorising the numerator and the denominator, we have

$$\lim_{x\to 3}\frac{x^2-9}{x^3-27} = \lim_{x\to 3}\frac{(x-3)(x+3)}{(x-3)(x^2+3x+9)} = \lim_{x\to 3}\frac{x+3}{x^2+3x+9} = \frac{3+3}{3^2+3\times 3+9} = \frac{2}{9}$$

Example 3. Find $\lim\limits_{x\to 2}\frac{2x^2+x-5}{3x+1}$.

Solution. Since $\lim\limits_{x\to 2}(3x+1) = 3\cdot 2+1 = 7\neq 0$

thus $$\lim_{x\to 2}\frac{2x^2+x-5}{3x+1} = \frac{\lim\limits_{x\to 2}(2x^2+x-5)}{\lim\limits_{x\to 2}(3x+1)} = \frac{2\times 2^2+2-5}{3\times 2+1} = \frac{5}{7}$$

Example 4. Find $\lim_{x \to 1} (3x^2 - 2x + 4)$.

Solution. $\lim_{x \to 1} (3x^2 - 2x + 4) = \lim_{x \to 1} 3x^2 - \lim_{x \to 1} 2x + \lim_{x \to 1} 4 = 3 \cdot 1^2 - 2 \cdot 1 + 4 = 5$

Example 5. Find $\lim_{x \to 0} \frac{x}{\sqrt{1+x} - 1}$.

Solution.

$$\lim_{x \to 0} \frac{x}{\sqrt{1+x} - 1} = \lim_{x \to 0} \frac{x\left(\sqrt{1+x} + 1\right)}{\left(\sqrt{1+x} - 1\right)\left(\sqrt{1+x} + 1\right)}$$

$$= \lim_{x \to 0} \frac{x\left(\sqrt{1+x} + 1\right)}{x} = \lim_{x \to 0}\left(\sqrt{1+x} + 1\right) = 2$$

Example 6. Find $\lim_{x \to \infty} \frac{3x^3 + x^2 + 4}{5x^3 - 3x - 7}$.

Solution. Dividing both the numerator and the denominator by x^3, we obtain

$$\lim_{x \to \infty} \frac{3x^3 + x^2 + 4}{5x^3 - 3x - 7} = \lim_{x \to \infty} \frac{3 + \frac{1}{x} + \frac{4}{x^3}}{5 - \frac{3}{x^2} - \frac{7}{x^3}} = \frac{3 + 0 + 0}{5 - 0 - 0} = \frac{3}{5}$$

Example 7. Find $\lim_{x \to \infty} \frac{x^5 - 4x^2}{6x^2 + 3x + 2}$.

Solution. Dividing both the numerator and the denominator by x^5, we obtain

$$\lim_{x \to \infty} \frac{x^5 - 4x^2}{6x^2 + 3x + 2} = \lim_{x \to \infty} \frac{1 - \frac{4}{x^3}}{\frac{6}{x^3} + \frac{3}{x^4} + \frac{2}{x^5}} = \frac{1 - 0}{0 + 0 + 0} = \infty$$

Because $\infty \notin R$, we say that the limit does not exist.

Example 8. Find $\lim_{x \to 0} \sin\frac{1}{x}$.

Solution. $\lim_{x \to 0} \sin\frac{1}{x} = \sin\frac{1}{0} = \sin\infty$

Because ∞ is dynamic, $\sin\infty$ takes values dynamically in the interval [-1, 1]. We say that it oscillates without limit. That is, the limit does not exist.

Example 9. Find $\lim_{x\to\infty}\frac{1}{x}\sin x$.

Solution. $\lim_{x\to\infty}\frac{1}{x}\sin x = \frac{1}{\infty}\sin\infty = 0\cdot\sin\infty = 0$

Example 10. Find $\lim_{n\to\infty}\frac{3n+1}{n}$.

Solution. $\lim_{n\to\infty}\frac{3n+1}{n} = \lim_{n\to\infty}(3+\frac{1}{n}) = 3+\frac{1}{\infty} = 3+0 = 3$

Example 11. Find $\lim_{n\to\infty}\frac{1+(-1)^n}{2}$.

Solution. $\lim_{n\to\infty}\frac{1+(-1)^n}{2} = \frac{1+(-1)^\infty}{2} = 0 \text{ or } 1$

Because ∞ is dynamic, the limit sometimes takes 0 and sometimes 1. We say that it oscillates without limit. That is, the limit does not exist.

2.3. Order of infinitesimals

Although all infinitesimals are variables that approach zero, the rates at which different infinitesimals approach zero are not necessarily the same and can differ significantly. For example, as $x \to 0$, the variables x, $2x$ and x^2 are all infinitesimals, but their rates of approaching zero are different. A comparison is shown in the table below.

x	1	0.5	0.1	0.01	0.001	$\cdots \to 0$
$2x$	2	1	0.2	0.02	0.002	$\cdots \to 0$
x^2	1	0.25	0.01	0.0001	0.000001	$\cdots \to 0$

Clearly, x^2 approaches zero much faster than x and $2x$. The rates are relative and are compared to each other. The following introduces the concept of the order of an infinitesimal by comparing the rates at which two infinitesimals approach zero.

Definition

Let α and β be two infinitesimals in the same process.

If $\lim \frac{\beta}{\alpha} = 0$, then we say that β is an infinitesimal of higher order than α, and we denote this by $\beta = o(\alpha)$.

If $\lim \frac{\beta}{\alpha} = c \neq 0$ (c is a constant), then we say that β and α are infinitesimals of the same order.

In the particular case where $c = 1$, we say that β and α are equivalent infinitesimals and denote it by $\beta \sim \alpha$.

When $c = \infty$, we say that β is an infinitesimal of lower order than α.

For example, since $\lim\limits_{x \to 0} \frac{x^2}{x} = \lim\limits_{x \to 0} x = 0$, it follows that as $x \to 0$, x^2 is an infinitesimal of higher order than x. We may denote this by $x^2 = o(x)$.

Conversely, since $\lim\limits_{x \to 0} \frac{x}{x^2} = \lim\limits_{x \to 0} \frac{1}{x} = \infty$, it follows that x is an infinitesimal of lower order than x^2.

For another example, since $\lim\limits_{x \to 0} \frac{x}{2x} = \lim\limits_{x \to 0} \frac{1}{2} = \frac{1}{2}$, it follows that as $x \to 0$, x and $2x$ are infinitesimals of the same order.

2.4. Two important limits

In the following, we will use two principles that limit exists to establish two important limits. With these two important limits, many problems in finding the limits of functions can be solved. And it also provides the necessary foundation for deriving the standard derivative formulae in the subsequent chapter.

a) Two principles that limit exists

Principle 1

Theorem. If, in the change process $x \to x_0$, the three variables $f(x)$, $g(x)$, and $h(x)$ always have a relationship

$$g(x) \le f(x) \le h(x)$$

and $$\lim_{x \to x_0} g(x) = \lim_{x \to x_0} h(x) = A$$

then $$\lim_{x \to x_0} f(x) = A$$

This theorem holds generally in mathematics. For example,

$$5 \le x \le 5 \xleftrightarrow{\text{definite} \quad \text{result}} x = 5$$

Principle 2

Consider a sequence $u_n = f(n)$. If, for all positive integers n, the inequality

$$f(n) < f(n+1)$$

always holds, then $f(n)$ is a monotonically increasing sequence.

If, for all positive integers n, the inequality

$$f(n) > f(n+1)$$

always holds, then $f(n)$ is a monotonically decreasing sequence.

If there exist two constants m and M (m < M) such that, for all positive integers n, the inequality

$$\mathrm{m} \le f(n) \le \mathrm{M}$$

always holds, then $f(n)$ is a bounded sequence.

Theorem. If a sequence $u_n = f(n)$ is monotonic and bounded, then the limit

$$\lim_{n \to \infty} f(n) = A$$

will definitely exist, where A is a constant.

b) Two important limits

(1) By applying principle 1, we can derive the first important limit.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Proof. Consider a unit circle, as shown in Figure 2.3.

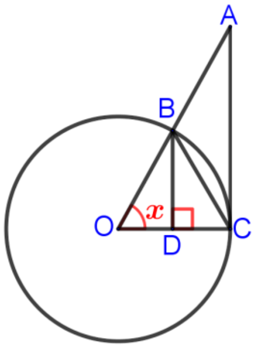

Figure 2.3

Let the central angle $B\hat{O}C = x \quad (0 < x < \frac{\pi}{2})$.

Then area of ΔOBC < area of sector OBC < area of ΔOAC

Since area of ΔOBC = $\frac{1}{2}$OC·BD = $\frac{1}{2}\cdot 1\cdot \sin x = \frac{1}{2}\sin x$

area of sector OBC = $\frac{1}{2}$OC·OB·$B\hat{O}C = \frac{1}{2}\cdot 1\cdot 1\cdot x = \frac{1}{2}x$

area of ΔOAC = $\frac{1}{2}$OC·AC = $\frac{1}{2}\cdot 1\cdot \tan x = \frac{1}{2}\tan x$

thus $\frac{1}{2}\sin x < \frac{1}{2}x < \frac{1}{2}\tan x$

Dividing the inequality by $\frac{1}{2}\sin x$, we obtain

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

That is, $\cos x < \frac{\sin x}{x} < 1$

Since the inequality holds when x is replaced by $-x$, it remains valid for all x in the interval $0 < |x| < \frac{\pi}{2}$.

And $\lim\limits_{x\to 0}\cos x = 1$, $\lim\limits_{x\to 0} 1 = 1$

By principle 1, we obtain the limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \qquad \square$$

Example 1. Find $\lim\limits_{x\to 0}\frac{\tan x}{x}$.

Solution. $\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\lim_{x\to 0}\frac{1}{\cos x} = 1$

Example 2. Find $\lim_{x\to 0}\frac{\tan 3x}{\sin 6x}$.

Solution. $\lim_{x\to 0}\frac{\tan 3x}{\sin 6x} = \lim_{x\to 0}\frac{\sin 3x}{\sin 6x\cos 3x} = \lim_{x\to 0}\frac{\sin 3x}{\sin 6x\cos 3x}\times\frac{18x}{18x}$

$$= \lim_{x\to 0}\frac{\sin 3x}{\sin 6x\cos 3x}\times\frac{6x}{3x}\times\frac{3}{6} = \lim_{x\to 0}(\frac{\sin 3x}{3x}\times\frac{6x}{\sin 6x}\times\frac{1}{\cos 3x}\times\frac{3}{6}) = \frac{3}{6} = \frac{1}{2}$$

Example 3. Find $\lim_{x\to 0}\frac{1-\cos x}{x^2}$.

Solution. $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{2^2\times\frac{x^2}{2^2}} = \frac{2}{4}\lim_{x\to 0}(\frac{\sin\frac{x}{2}}{\frac{x}{2}})^2 = \frac{1}{2}$

(2) By applying principle 2, we can derive the second important limit.

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

where e is a constant. It is an irrational number, and its value is

$$e = 2.7182818284590\ldots.$$

Proof. Let $f(n) = (1+\frac{1}{n})^n$.

First, we show that $f(n)$ is monotonically increasing. Applying the binomial theorem, we obtain

$$f(n) = (1+\frac{1}{n})^n$$

$$= 1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{3!}\cdot\frac{1}{n^3}+\cdots+\frac{n(n-1)(n-2)\cdots 2\cdot 1}{n!}\cdot\frac{1}{n^n}$$

$$= 1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{n!}(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{n-1}{n})$$

Similarly, we obtain

$$f(n+1) = (1+\frac{1}{n+1})^{n+1}$$

$$=1+1+\frac{1}{2!}(1-\frac{1}{n+1})+\frac{1}{3!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})+\cdots$$

$$+\frac{1}{n!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n-1}{n+1})+\frac{1}{(n+1)!}(1-\frac{1}{n+1})(1-\frac{2}{n+1})\cdots(1-\frac{n}{n+1})$$

Comparing the two expansions, the first two terms are identical. However, from the third term onwards, each term in $f(n+1)$ is greater than the corresponding term in $f(n)$. Furthermore, the expansion of $f(n+1)$ contains one more positive term at the end. Therefore,

$$f(n)<f(n+1) \qquad (n = 1, 2, 3, \ldots)$$

$f(n)$ is a monotonically increasing sequence.

In the following, we show that $f(n)$ is bounded. In the expansion of $f(n)$, all numbers in the brackets are less than 1. Consequently

$$f(n) < 1 + 1 + \frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!}+\ldots+\frac{1}{n!}$$

Since $\quad \frac{1}{2^{k-1}} \geq \frac{1}{k!} \qquad (k \geq 2)$

thus $\quad f(n)< 1 + 1 + \frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{k-2}}+\ldots+\frac{1}{2^{n-1}}=1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}=3-\frac{1}{2^{n-1}}<3$

That is, $\quad f(n)<3$

It can be seen from here that no matter how big the value n is, $f(n)$ is always less than 3. That is, $f(n)$ is a bounded sequence. Therefore, by principle 2, we obtain the limit.

$$\lim_{n\to\infty}(1+\frac{1}{n})^n=e$$

It can be shown that for a continuous independent variable x, we also have

$$\lim_{x\to\infty}(1+\frac{1}{x})^x=e$$

This formula can also be written in the form

$$\lim_{a\to 0}(1+a)^{\frac{1}{a}}=e$$

A logarithm with base e is called a natural logarithm. The natural logarithm of x is denoted by $\ln x$. In mathematics, we frequently employ the natural logarithmic functions, $y = \ln x$ and exponential functions with base e, $y = e^x$.

Example 4. Find $\lim_{x\to\infty}(\frac{x+4}{x})^x$.

Solution. $\lim_{x\to\infty}(\frac{x+4}{x})^x = \lim_{x\to\infty}(1+\frac{4}{x})^x = \lim_{x\to\infty}(1+\frac{1}{(\frac{x}{4})})^{x\cdot\frac{4}{4}} = \lim_{x\to\infty}[(1+\frac{1}{(\frac{x}{4})})^{\frac{x}{4}}]^4 = e^4$

Example 5. Find $\lim_{x\to\infty}(1-\frac{1}{x})^x$.

Solution. $\lim_{x\to\infty}(1-\frac{1}{x})^x = \lim_{x\to\infty}[(1+\frac{1}{(-x)})^{-x}]^{-1} = e^{-1}$

Exercise 2

1. Find the following limits.

a. $\lim_{x\to 3}\left(5+x^2-x\right)$

b. $\lim_{x\to 4}\frac{x^2-16}{x-4}$

c. $\lim_{x\to 1}\frac{x-1}{\sqrt{x+1}}$

d. $\lim_{x\to 1}\frac{x+1}{x^2-x-2}$

e. $\lim_{x\to 2}\frac{\sqrt{x+7}-3}{x-2}$

f. $\lim_{x\to\infty}\frac{x^3+x^2-4x+1}{2x^3+x+9}$

g. $\lim_{x\to 3}\frac{x^2-2x-3}{x-3}$

h. $\lim_{x\to\infty}\frac{2x+5}{x}$

i. $\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}$

j. $\lim_{x\to 0}\sin(x+\pi)$

k. $\lim_{x\to -5}(3x+14)^{99}$

l. $\lim_{x\to -3}4$

2. Find the following limits.

a. $\lim_{x\to -1}\sqrt{8-x}$

b. $\lim_{x\to 2}\sqrt{x^2+12}$

c. $\lim\limits_{x\to4}\dfrac{\sqrt{x}-2}{x-2}$

d. $\lim\limits_{x\to2}\dfrac{x^3-8}{x-2}$

e. $\lim\limits_{x\to9}\dfrac{9-x}{3-\sqrt{x}}$

f. $\lim\limits_{x\to0}\dfrac{4x}{(3+x)^2-9}$

g. $\lim\limits_{x\to-1}\dfrac{x^2+5x+6}{x^2+3x+2}$

h. $\lim\limits_{x\to1}\dfrac{2x^2-10x+8}{3x^2-3x}$

3. Find the following limits.

a. $\lim\limits_{x\to3}\dfrac{x^4-81}{x^3-3x^2}$

b. $\lim\limits_{x\to3}\dfrac{x^3-3x^2+12x-4}{x^3-4x}$

c. $\lim\limits_{x\to2}\dfrac{x-2}{\sqrt{x+2}-2}$

d. $\lim\limits_{x\to\infty}\dfrac{3x+1}{3x^2+x-4}$

e. $\lim\limits_{x\to-2}\dfrac{x^2+3x+2}{x^2-x-6}$

f. $\lim\limits_{x\to-3}\dfrac{\sin\dfrac{\pi x}{6}}{x+1}$

g. $\lim\limits_{x\to0}\dfrac{\sqrt{x+4}-2}{x}$

h. $\lim\limits_{x\to-8}\dfrac{\sqrt{1-x}-3}{2+\sqrt[3]{x}}$

4. Find the following limits.

a. $\lim\limits_{x\to\infty}\dfrac{x^5-x^4+3}{4x^5-3x^3+2x+6}$

b. $\lim\limits_{x\to3}\dfrac{x^3-3x^2+x-3}{x-3}$

c. $\lim\limits_{x\to1}\dfrac{x^2-1}{(x-1)\sqrt{x^2+1}}$

d. $\lim\limits_{x\to4}\dfrac{2-\sqrt{x}}{3-\sqrt{2x+1}}$

e. $\lim\limits_{x\to3}\dfrac{2x^3+5x^2-39x+18}{x^3+3x^2-13x-15}$

f. $\lim\limits_{x\to-1}\dfrac{x^3+6x^2+11x+6}{x^3-4x^2-x+4}$

5. Find the following limits.

a. $\lim\limits_{x\to0}\dfrac{\sin 2x}{\sin 5x}$

b. $\lim\limits_{x\to\infty}\left(1+\dfrac{3}{x}\right)^{3x}$

c. $\lim\limits_{x\to0}\dfrac{\tan 3x}{\tan x}$

d. $\lim\limits_{x\to0}\dfrac{x}{\sin 5x}$

e. $\lim\limits_{x\to0}\dfrac{1-\cos 4x}{x^2}$

f. $\lim\limits_{x\to\infty}\left(\dfrac{5x+2}{5x}\right)^{x}$

g. $\lim\limits_{x\to0}(1-2x)^{\frac{1}{x}}$

h. $\lim\limits_{x\to0}\dfrac{\tan^3 x-\sin^3 x}{x^3}$

Content Introduction

This book is suitable for university students, college students, mathematics enthusiasts interested in calculus, and mathematics teachers. It comprises 16 chapters and is divided into two volumes. Volume 1 covers preparatory knowledge, limit theory, calculus of single variable functions, first order and higher order differential equations. Volume 2 covers infinite series, vector algebra and space analytic geometry, multivariable calculus, Fourier series and transform.

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