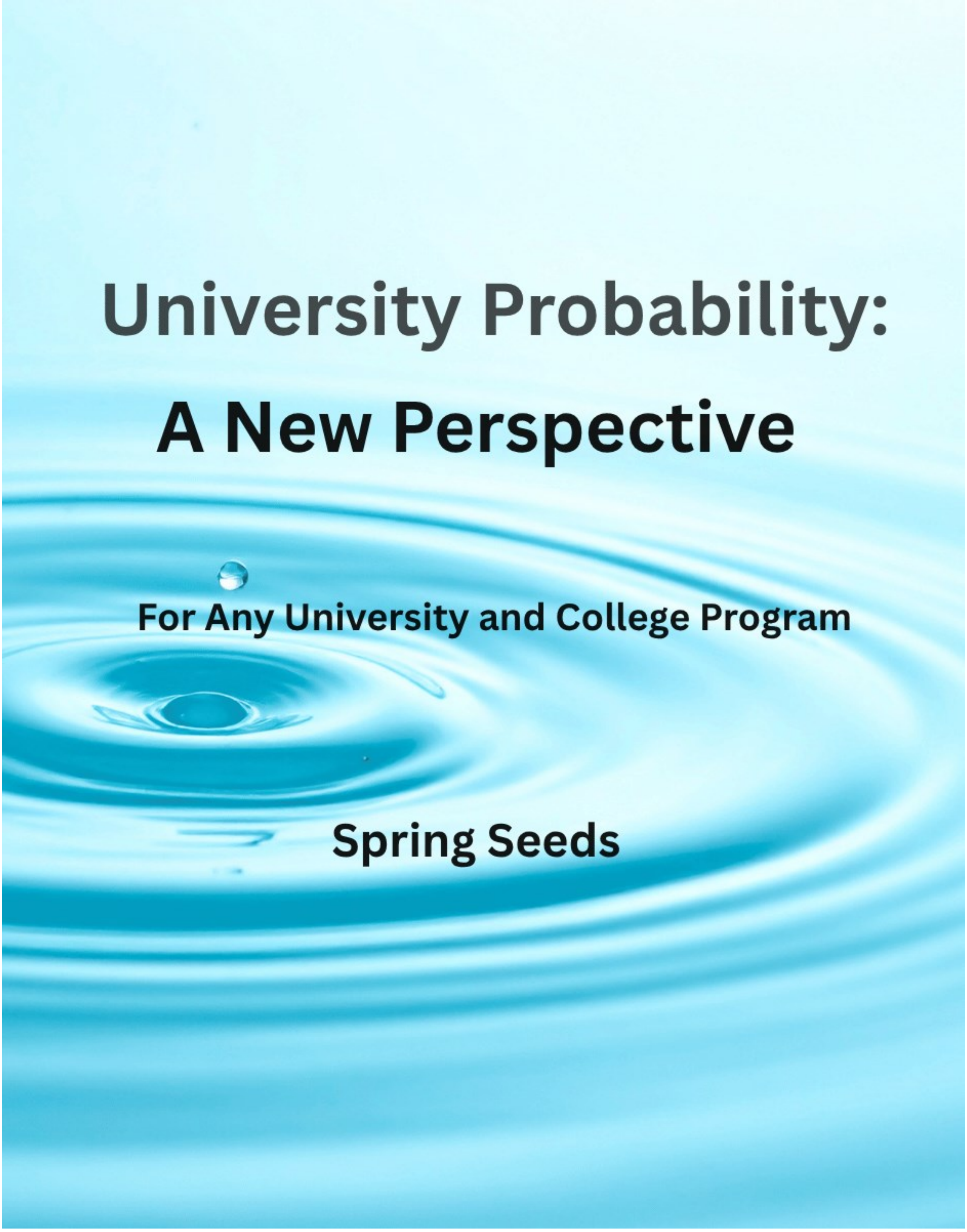

# University Probability:

# A New Perspective

**For Any University and College Program**

**Spring Seeds**

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# Preface

Probability theory is the branch of mathematics dedicated to studying and revealing the nature of random phenomena. It is also a core discipline for cultivating students' careful and rigorous thinking and enhancing overall mathematical skills. In the education of probability theory, mastering foundational concepts and principles is of paramount importance, serving as a necessary pathway to a comprehensive understanding of the subject.

Having read and studied numerous probability theories, we observed that they lack specificity and simplicity when elaborating on these concepts and principles of probability. To fill this educational gap and better meet students' needs, we developed this textbook.

This textbook has been successfully implemented in the teaching of mathematics courses, where it received high praise from students. Students note the concepts and principles of this textbook are presented with exceptional clarity, concrete detail, and strong logical cohesion, making the knowledge highly accessible. Furthermore, students have consistently achieved excellent academic results in the study of probability.

Through extensive teaching practice, this textbook has proven to address highly effectively the traditional difficulties associated with learning probability theory, transforming it into an easy-to-understand and engaging subject.

To ensure maximum adaptability, it contains content applicable to different university and college programmes, which can be used by various university and college courses.

Spring Seeds

2026

# Chapter 1 Permutation and Combination

Knowledge of permutations and combinations is the computational basis of probability theory. Permutation and combination study the selection of $r$ elements from $n$ elements ($r \le n$) to form ordered permutations or unordered combinations.

## 1.1. Permutation

To illustrate clearly the concept of permutation, let us look at two examples first.

**Example 1**. To hold a meeting, choose a chairperson and a secretary from four people: a, b, c, and d. How many different ways are there to choose (that is, what is the number of permutations)?

**Solution**. There are four choices for the chairperson. After this, there are three choices for the secretary from the remaining three people.

Hence, the number of permutations is $\quad {}^{n}P_{r} = {}^{4}P_{2} = 4\cdot(4-1) = 12$

The symbol ${}^{n}P_{r}$ represents the number of permutations when $r$ elements are chosen at random from $n$ different elements ($r \le n$) to form ordered permutations.

Here, we list all possible outcomes of the ordered permutations in this example (expressed by a set).

$$\begin{aligned} A &= \{(x_1, x_2)\} \\ &= \{(a, b), (a, c), (a, d), (b, a), (b, c), (b, d), (c, a), (c, b), (c, d), (d, a), (d, b), (d, c)\} \end{aligned}$$

The elements $(x_1, x_2, \ldots, x_n)$ in this a set are called ordered $n$-tuples.

Except for $x_1 = x_2 = \ldots = x_n$

$$(x_1, x_2, \ldots, x_n) \ne (x_2, x_1, \ldots, x_n) \ne \ldots \ne (x_n, \ldots, x_2, x_1)$$

Therefore, in this example, (a, b) ≠ (b, a), …, (c, d) ≠ (d, c).

The number of elements in set A is $n(\mathrm{A}) = 12$.

So $$^4P_2 = n(\mathrm{A}) = 4\cdot(4-1) = 12$$

**Example 2**. There are five elements in set A, where A = {1, 2, 3, 4, 5}. Calculate

- **a.** the number of permutations when 3 elements are chosen from set A to form ordered permutations.
- **b.** the number of permutations when 5 elements are chosen from set A to form ordered permutations.

**Solution**.

**a.** We know that $n = n(\mathrm{A}) = 5$ and $r = 3$.

So $$^nP_r = {}^5P_3 = 5 \cdot (5-1)(5-2) = 5 \cdot 4 \cdot 3 = 60$$

**b.** We know that $n = n(\mathrm{A}) = 5$ and $r = 5$.

So $$^nP_r = {}^5P_5 = 5 \cdot (5-1)(5-2)(5-3)(5-4) = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$$

Here, $^5P_5$ is called a **complete permutation** or **5 factorial**, denoted as $^5P_5 = 5!$.

That is, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

In the same way, we can calculate ***n*** **factorial**: $n! = {}^nP_n$

The $n$ factorial formula is $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1 \qquad (n \in \mathrm{Z}^+)$

$0! = 1$

By deriving from the above examples, we obtain the formula for calculating the number of permutations.

$$^nP_r = \frac{n!}{(n-r)!} = \underbrace{n(n-1)(n-2)\cdots(n-(r-1))}_{r \quad factors} \qquad (0 \le r \le n)$$

## 1.2. Combination

Below, we illustrate the concept of combinations using two examples.

**Example 1**. There are 5 different elements in set A, where A = {1, 2, 3, 4, 5}. Find the number of combinations when 3 elements are chosen from set A to form unordered combinations.

**Solution**. We know that $n = n(\mathrm{A}) = 5$ and $r = 3$.

So the number of combinations $= \binom{n}{r} = \binom{5}{3} = \dfrac{5\times4\times3}{3\times2\times1} = 10$

The symbol $\binom{n}{r}$ means the number of combinations when $r$ elements are chosen from $n$ different elements ($r \le n$) to form unordered combinations.

Here, we list all possible outcomes of the unordered combinations in this example (expressed by a set).

$$\begin{aligned}\mathrm{B} &= \{\{x_1, x_2, x_3\}\}\\ &= \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 2, 5\}, \{1, 3, 4\}, \{1, 3, 5\}, \{1, 4, 5\}, \{2, 3, 4\},\\ &\qquad \{2, 3, 5\}, \{2, 4, 5\}, \{3, 4, 5\}\}\end{aligned}$$

For unordered combinations, it is always true that

$$\{x_1, x_2, \ldots, x_n\} = \{x_2, x_1, \ldots, x_n\} = \ldots = \{x_n, \ldots, x_2, x_1\}$$

So, in this example,

$$\{x_1, x_2, x_3\} = \{x_2, x_1, x_3\} = \ldots = \{x_3, x_2, x_1\}$$

For example $\{1, 2, 3\} = \{1, 3, 2\} = \{2, 1, 3\} = \{2, 3, 1\} = \{3, 1, 2\} = \{3, 2, 1\}$

Hence, the number of elements in set B is $n(\mathrm{B}) = 10$.

So $$\binom{5}{3} = n(\mathrm{B}) = \frac{5\times4\times3}{3\times2\times1} = 10$$

**Example 2**. Five cards are drawn from a pack of 52 playing cards. How many possible selections are there (that is, what is the number of combinations)?

**Solution**. It is given that $n = 52$ and $r = 5$.

So $$\binom{n}{r} = \binom{52}{5} = \frac{52\times51\times50\times49\times48}{5\times4\times3\times2\times1} = 2598960$$

By deriving from the above examples, we can obtain the formula for the number of combinations.

$$\binom{n}{r} = \frac{\overbrace{n(n-1)(n-2)\cdots(n-(r-1))}^{r\ \ factors}}{r!} = \frac{n!}{r!(n-r)!} \qquad \text{where} \quad 0 \le r \le n$$

When $r = 0$, $\binom{n}{0} = 1$.

When $r = n$, $\binom{n}{n} = 1$.

## 1.3. Examples

**Example 1**. 5 boys and 4 girls sit in a row. How many ways can this be arranged

- **a.** if there is no restriction?
- **b.** if the boys and girls each sit together?

**Solution**.

**a.** Since there is no restriction, we will put 9 people in permutation. Thus, we have

$$^9P_9 = 9! = 362880 \quad \text{ways.}$$

**b.** If the boys and girls each sit together, we need to first treat all boys as one big boy and all girls as one big girl, then there are 2! ways to arrange them. Then there are 5! ways to arrange all the boys and 4! ways to arrange all the girls. Thus, altogether, there are

$$2!5!4! = 5760 \quad \text{ways.}$$

**Example 2**. How many different ways are there to arrange all the letters from the word COMMITTEE?

**Solution**. There are altogether 9 letters in this word, so arranging all 9 letters in permutation, we have $^9P_9 = 9! = 362880$ ways. However, there are two letters M, which will give the same permutations. We need to exclude this so $\frac{9!}{2!}$. We also have two letters T and two letters E, thus altogether, we have

$$\frac{9!}{2!2!2!} = 45360 \quad \text{different ways.}$$

**Example 3**. Six flags of different colours can be hung in a vertical line to form different signals.

- **a.** How many different signals can be formed if all 6 coloured flags are used?
- **b.** How many different signals can be formed if any number of coloured flags are used?

**Solution**. **a.** If all 6 coloured flags are used, we will put all 6 colours in permutation, so we have

$$^6P_6 = 6! = 720 \quad \text{different signals.}$$

**b.** If any number of coloured flags can be used, we can choose 1 colour to form a signal, and then we have ${}^6P_1$ signals. Or we can choose 2 colours to form a signal, and then we have ${}^6P_2$ signals. Similarly, if 3, 4, 5 or 6 colours are chosen, we will have ${}^6P_3$, ${}^6P_4$, ${}^6P_5$ and ${}^6P_6$ signals respectively. Thus, altogether, we have

$${}^6P_1 + {}^6P_2 + {}^6P_3 + {}^6P_4 + {}^6P_5 + {}^6P_6 = 1956 \qquad \text{different signals.}$$

**Example 4**.

**a.** How many ways can a committee of 3 men and 2 women be chosen from 7 men and 5 women?

**b.** Among the 12 people, a divorced couple do not want to be chosen together. How many ways can the committee be formed?

**Solution**. **a.** We need to choose 3 men from 7 men, which can be done in $\binom{7}{3}$ ways. We need to choose 2 women from 5 women, which can be done in $\binom{5}{2}$ ways. Thus, altogether, there are

$$\binom{7}{3}\binom{5}{2} = 350 \qquad \text{ways.}$$

**b. Method 1** There are three cases in which the couple are not chosen simultaneously.

Case 1. Neither member of the couple is chosen, so we need to choose 3 men from the remaining 6 men and 2 women from the remaining 4 women, and there are $\binom{6}{3}\binom{4}{2}$ ways.

Case 2. The husband is chosen and the wife is not chosen, so we need to choose 2 men from the remaining 6 men and 2 women from the remaining 4 women. Thus, we have $\binom{6}{2}\binom{4}{2}$ ways.

Case 3. The husband is not chosen and the wife is chosen, so we need to choose 3 men from the remaining 6 men and 1 woman from the remaining 4 women. Thus, we have $\binom{6}{3}\binom{4}{1}$ ways.

Adding all these three cases together, we have

$$\binom{6}{3}\binom{4}{2}+\binom{6}{2}\binom{4}{2}+\binom{6}{3}\binom{4}{1}=290 \quad \text{ways.}$$

**Method 2** If the couple are both chosen, we need to choose 2 men from the remaining 6 men and 1 woman from the remaining 4 women. Thus, we have $\binom{6}{2}\binom{4}{1}$ ways. Subtracting this from the total number of ways with no restriction (obtained in **a.**) will give us the number of ways that the couple are not both chosen.

$$\binom{7}{3}\binom{5}{2}-\binom{6}{2}\binom{4}{1}=290$$

**Example 5**. There are 11 boys and 20 girls in a debate club. The club must choose a team of 4 people for a debate conference. How many different conference teams are possible

- **a.** if the team consists of 2 boys and 2 girls?
- **b.** if the team consists of more girls than boys?
- **c.** if the team consists of people of the same gender?

**Solution**. **a.** If the team consists of 2 boys and 2 girls, there are $\binom{11}{2}$ ways to choose 2 boys from 11 boys. And there are $\binom{20}{2}$ ways to choose 2 girls from 20 girls. Thus, altogether, there are

$$\binom{11}{2}\binom{20}{2} = 10450 \qquad \text{teams.}$$

**b.** If the team consists of more girls than boys, then it could have 3 girls and 1 boy or 4 girls only. So, there are altogether

$$\binom{20}{3}\binom{11}{1} + \binom{20}{4} = 17385 \quad \text{teams.}$$

**c.** If the team consists of people of the same gender, then it could be either 4 boys only or 4 girls only. So altogether, we have

$$\binom{11}{4} + \binom{20}{4} = 5175 \qquad \text{teams.}$$

**Exercise 1**

**1**. A hotel director needs to select a manager, an assistant manager and a restaurant manager from a list of 12 people. How many ways can this be done?

**2**. The number plates in one country have 3 letters from a 26-letter alphabet followed by 3 digits from 0 to 9.

- **a.** How many number plates can be created if there are no other restrictions?
- **b.** What if letters cannot be repeated?

**c.** What if digits cannot be repeated?

**d.** What if both letters and digits cannot be repeated?

**3**. At a dog show, the judge must choose first, second and third place finishers from a group of 28 dogs. How many ways can this be done?

**4**. A class of 30 students is going on a field trip. The first bus has only 15 seats, so the teacher has to choose 15 students to take the first bus. The remaining students and the teacher wait for the second bus.

**a.** If there is no restriction, how many ways are there to choose 15 students?

**b.** If three students are friends and they want to travel on the same bus, how many ways are there to choose 15 students?

**5**. A teacher starts each lesson by choosing 3 students to present solutions to homework problems to the class. If there are 30 students in the class

**a.** how many ways can the teacher make the selection?

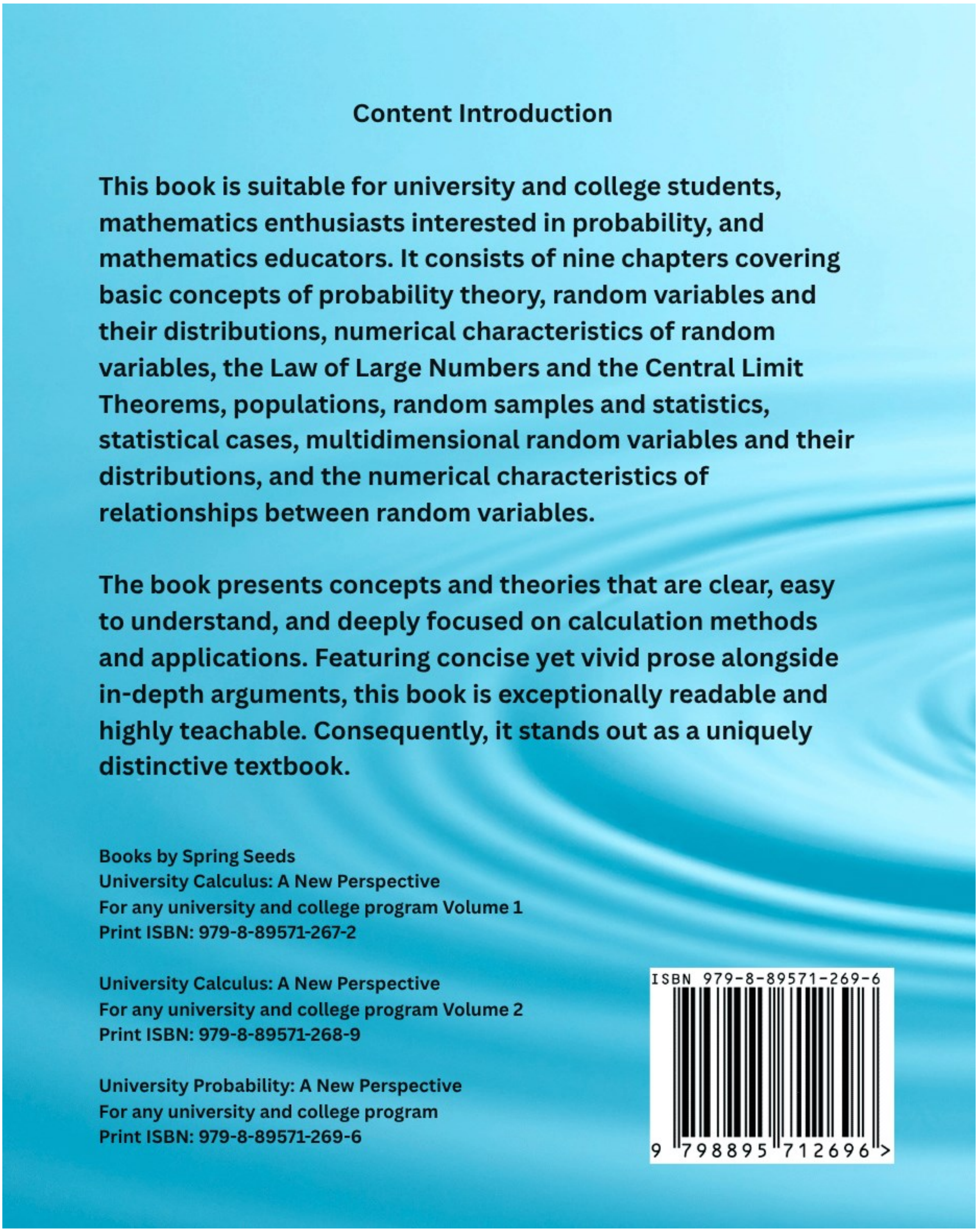
**b.** Anton and Maya presented during the last lesson. If the teacher wants to select other students, how many ways can the teacher make the selection?

**6**. A jury consists of 7 people selected from 10 male and 10 female candidates.

**a.** How many different ways of selection are there?

**b.** Out of the 20 candidates, two are from the same family and cannot be selected simultaneously. How many different ways of selection are there?

**c.** If the jury must have at least 3 people from each gender, how many different ways of selection are there?



Content Introduction

This book is suitable for university and college students, mathematics enthusiasts interested in probability, and mathematics educators. It consists of nine chapters covering basic concepts of probability theory, random variables and their distributions, numerical characteristics of random variables, the Law of Large Numbers and the Central Limit Theorems, populations, random samples and statistics, statistical cases, multidimensional random variables and their distributions, and the numerical characteristics of relationships between random variables.

The book presents concepts and theories that are clear, easy to understand, and deeply focused on calculation methods and applications. Featuring concise yet vivid prose alongside in-depth arguments, this book is exceptionally readable and highly teachable. Consequently, it stands out as a uniquely distinctive textbook.

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