

## Chapter 2 Limits

### 2.1. Limits of functions and variables

The concept and theory of limits are mathematical theories from a dynamic point of view. They form the foundation of calculus. The two fundamental concepts in calculus, differentiation and integration, are based on the concept of the limit.

The approach symbol  $x \rightarrow x_0$  indicates the change process of the independent variable  $x$  approaching the limit value  $x_0$ .

from the left of  $x_0$   $x \rightarrow x_0$  Or  $x_0 \leftarrow x$  from the right of  $x_0$

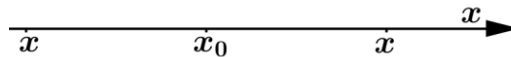


Figure 2.1

As shown in Figure 2.1, at the beginning,  $x$  takes value in the direction of  $x_0$ . In the end,  $x$  takes a value that is infinitely close to  $x_0$ . That is, at the end of the change process,

$$x = x_0 + a \quad \text{or} \quad x \approx x_0$$

where  $a$  is an infinitesimal.

An infinitesimal  $a$  is a variable that approaches zero, that is,  $a \rightarrow 0$ . At the end of the change process

$$a \approx 0 \quad \Leftrightarrow \quad 0 < |a| \leq 0.05$$

Using the above notion of the dynamic approach, we now provide an equally dynamic definition for the limit of a function:

If a function  $f(x)$  is defined in a neighbourhood of  $x_0$ , when  $x$  approaches infinitely close to  $x_0$ , the corresponding functional value  $f(x)$  will approach infinitely close to a certain constant  $A$ , then  $A$  is called the limit of the function  $f(x)$  as  $x \rightarrow x_0$ , denoted by

$$\lim_{x \rightarrow x_0} f(x) = A$$

If  $f(x)$  is a polynomial function or a fractional function whose denominator's limit is not zero as  $x \rightarrow x_0$ , then we have

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

The limit symbol,  $\lim_{x \rightarrow x_0} (\bullet)$ , means that the independent variable  $x$  in the expression  $(\bullet)$  will finally take the value of  $x = x_0$  in the change process when  $x$  approaches  $x_0$ .

For the change process of  $x$  as  $x \rightarrow x_0$ , the corresponding change process of  $f(x)$  is  $f(x) \rightarrow A$ , this is generally denoted by

$$f(x) \rightarrow A \quad (x \rightarrow x_0)$$

**In summary**

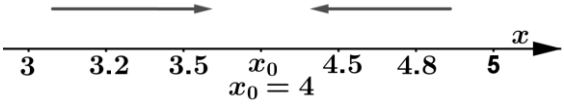


Figure 2.2

The approach symbol  $x \rightarrow x_0$  and the limit symbol  $\lim_{x \rightarrow x_0} (\bullet)$  mean that at the beginning,  $x$  takes value in the direction of  $x_0$ , either from the left of  $x_0$  ( $x < x_0$ ) or from the right of  $x_0$  ( $x > x_0$ ), as shown in Figure 2.2.

$$x = 3 \dots 3.2 \dots 3.5 \dots x_0 - a, x_0$$

or

$$x = 5 \dots 4.8 \dots 4.5 \dots x_0 + a, x_0$$

At the end of the change process

$$x = x_0 \pm a \approx x_0 \quad (\text{for } x \rightarrow x_0)$$

$$x = x_0 \quad (\text{for } \lim_{x \rightarrow x_0} (\bullet))$$

With the above concept of limits, we can now apply the limit methods to prove the

indeterminate form  $\frac{0}{0} = a$ .

**Proof.** Let  $\lim_{x \rightarrow 0} x = 0$

then  $a \cdot \lim_{x \rightarrow 0} x = a \cdot 0 = 0$   $a$  is a random number.

Dividing both sides by  $\lim_{x \rightarrow 0} x$  gives

$$a \cdot \frac{\lim_{x \rightarrow 0} x}{\lim_{x \rightarrow 0} x} = \frac{0}{\lim_{x \rightarrow 0} x}$$

That is,  $a \cdot 1 = \frac{0}{0}$

Hence,  $a = \frac{0}{0} = 0 \cdot \infty = \frac{\infty}{\infty} = \infty - \infty$  □

**Note:** Above, we have given the concepts and definitions of approach and limit from a dynamic point of view. They should be easy to understand and grasp. However, in order to have a more comprehensive understanding of the nature of the limit, we will provide the definition of the limit from a static point of view below:

If, for any given positive number  $\varepsilon > 0$ , there always exists a positive number  $\delta > 0$  such that whenever  $0 < |x - x_0| < \delta$ , the inequality

$$|f(x) - A| < \varepsilon$$

always holds, then the function  $f(x)$  is said to have a limit at constant  $A$  when  $x$  approaches  $x_0$ . This is denoted by

$$\lim_{x \rightarrow x_0} f(x) = A \quad \text{or} \quad f(x) \rightarrow A \quad (x \rightarrow x_0)$$

Here, the limit and approach have the same definition. However, their properties and calculation results are different. Care must be taken to distinguish the differences and do not confuse them. Furthermore, this definition only conforms to mathematical logic but lacks operability and is difficult to understand.  $\square$

## 2.2. Computation rules of limits

As  $x \rightarrow x_0$ , the limits of the functions  $f(x)$  and  $g(x)$  are  $A$  and  $B$ , respectively.

That is,  $\lim_{x \rightarrow x_0} f(x) = A$        $\lim_{x \rightarrow x_0} g(x) = B$

Then (1)  $\lim_{x \rightarrow x_0} [f(x) \pm g(x)] = \lim_{x \rightarrow x_0} f(x) \pm \lim_{x \rightarrow x_0} g(x) = A \pm B$

(2)  $\lim_{x \rightarrow x_0} [f(x) \times g(x)] = \lim_{x \rightarrow x_0} f(x) \times \lim_{x \rightarrow x_0} g(x) = A \times B$

If  $g(x) = b$  is a constant, then

$$\lim_{x \rightarrow x_0} bf(x) = b \lim_{x \rightarrow x_0} f(x) = bA$$

(3)  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow x_0} f(x)}{\lim_{x \rightarrow x_0} g(x)} = \frac{A}{B} \quad B \neq 0$

- In the calculations above, the variable  $x$  is a continuous variable. For a discontinuous integer variable  $n$ , except  $n \rightarrow \infty$ ,  $\lim_{n \rightarrow \infty} (\bullet)$ , there is no other computation of limit.

**Example 1.** Find  $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ .

**Solution.** As  $x \rightarrow 2$ ,  $x \neq 2$ , the common factor  $(x - 2)$  can be cancelled from the numerator and the denominator.

$$\text{Thus } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x + 2)(x - 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$$

(At the end of the change process,  $x = 2$ .)

**Example 2.** Find  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^3 - 27}$ .

**Solution.** Because  $\lim_{x \rightarrow 3} (x^3 - 27) = 0$ , the computation rule (3) cannot be used directly.

However, as  $x \rightarrow 3$ ,  $x \neq 3$ . Both the numerator and denominator have a factor  $(x - 3)$ .

Factorising the numerator and the denominator, we have

$$\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^3 - 27} = \lim_{x \rightarrow 3} \frac{(x - 3)(x + 3)}{(x - 3)(x^2 + 3x + 9)} = \lim_{x \rightarrow 3} \frac{x + 3}{x^2 + 3x + 9} = \frac{3 + 3}{3^2 + 3 \times 3 + 9} = \frac{2}{9}$$

**Example 3.** Find  $\lim_{x \rightarrow 2} \frac{2x^2 + x - 5}{3x + 1}$ .

**Solution.** Since  $\lim_{x \rightarrow 2} (3x + 1) = 3 \cdot 2 + 1 = 7 \neq 0$

$$\text{thus } \lim_{x \rightarrow 2} \frac{2x^2 + x - 5}{3x + 1} = \frac{\lim_{x \rightarrow 2} (2x^2 + x - 5)}{\lim_{x \rightarrow 2} (3x + 1)} = \frac{2 \times 2^2 + 2 - 5}{3 \times 2 + 1} = \frac{5}{7}$$

**Example 4.** Find  $\lim_{x \rightarrow 1} (3x^2 - 2x + 4)$ .

**Solution.**  $\lim_{x \rightarrow 1} (3x^2 - 2x + 4) = \lim_{x \rightarrow 1} 3x^2 - \lim_{x \rightarrow 1} 2x + \lim_{x \rightarrow 1} 4 = 3 \cdot 1^2 - 2 \cdot 1 + 4 = 5$

**Example 5.** Find  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1}$ .

**Solution.** 
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x}{\sqrt{1+x}-1} &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+x}+1)}{(\sqrt{1+x}-1)(\sqrt{1+x}+1)} \\ &= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+x}+1)}{x} = \lim_{x \rightarrow 0} (\sqrt{1+x}+1) = 2 \end{aligned}$$

**Example 6.** Find  $\lim_{x \rightarrow \infty} \frac{3x^3 + x^2 + 4}{5x^3 - 3x - 7}$ .

**Solution.** Dividing both the numerator and the denominator by  $x^3$ , we obtain

$$\lim_{x \rightarrow \infty} \frac{3x^3 + x^2 + 4}{5x^3 - 3x - 7} = \lim_{x \rightarrow \infty} \frac{3 + \frac{1}{x} + \frac{4}{x^3}}{5 - \frac{3}{x^2} - \frac{7}{x^3}} = \frac{3+0+0}{5-0-0} = \frac{3}{5}$$

**Example 7.** Find  $\lim_{x \rightarrow \infty} \frac{x^5 - 4x^2}{6x^2 + 3x + 2}$ .

**Solution.** Dividing both the numerator and the denominator by  $x^5$ , we obtain

$$\lim_{x \rightarrow \infty} \frac{x^5 - 4x^2}{6x^2 + 3x + 2} = \lim_{x \rightarrow \infty} \frac{1 - \frac{4}{x^3}}{\frac{6}{x^3} + \frac{3}{x^4} + \frac{2}{x^5}} = \frac{1-0}{0+0+0} = \infty$$

Because  $\infty \notin \mathbb{R}$ , we say that the limit does not exist.

**Example 8.** Find  $\lim_{x \rightarrow 0} \sin \frac{1}{x}$ .

**Solution.**  $\lim_{x \rightarrow 0} \sin \frac{1}{x} = \sin \frac{1}{0} = \sin \infty$

Because  $\infty$  is dynamic,  $\sin \infty$  takes values dynamically in the interval  $[-1, 1]$ . We say that it oscillates without limit. That is, the limit does not exist.

**Example 9.** Find  $\lim_{x \rightarrow \infty} \frac{1}{x} \sin x$ .

**Solution.**  $\lim_{x \rightarrow \infty} \frac{1}{x} \sin x = \frac{1}{\infty} \sin \infty = 0 \cdot \sin \infty = 0$

**Example 10.** Find  $\lim_{n \rightarrow \infty} \frac{3n+1}{n}$ .

**Solution.**  $\lim_{n \rightarrow \infty} \frac{3n+1}{n} = \lim_{n \rightarrow \infty} \left(3 + \frac{1}{n}\right) = 3 + \frac{1}{\infty} = 3 + 0 = 3$

**Example 11.** Find  $\lim_{n \rightarrow \infty} \frac{1+(-1)^n}{2}$ .

**Solution.**  $\lim_{n \rightarrow \infty} \frac{1+(-1)^n}{2} = \frac{1+(-1)^\infty}{2} = 0 \text{ or } 1$

Because  $\infty$  is dynamic, the limit sometimes takes 0 and sometimes 1. We say that it oscillates without limit. That is, the limit does not exist.

### 2.3. Order of infinitesimals

Although all infinitesimals are variables that approach zero, the rates at which different infinitesimals approach zero are not necessarily the same and can differ significantly. For example, as  $x \rightarrow 0$ , the variables  $x$ ,  $2x$  and  $x^2$  are all infinitesimals, but their rates of approaching zero are different. A comparison is shown in the table below.

$x$	1	0.5	0.1	0.01	0.001	...	$\rightarrow 0$
$2x$	2	1	0.2	0.02	0.002	...	$\rightarrow 0$
$x^2$	1	0.25	0.01	0.0001	0.000001	...	$\rightarrow 0$

Clearly,  $x^2$  approaches zero much faster than  $x$  and  $2x$ . The rates are relative and are compared to each other. The following introduces the concept of the order of an infinitesimal by comparing the rates at which two infinitesimals approach zero.

### Definition

Let  $\alpha$  and  $\beta$  be two infinitesimals in the same process.

If  $\lim \frac{\beta}{\alpha} = 0$ , then we say that  $\beta$  is an infinitesimal of higher order than  $\alpha$ , and we

denote this by  $\beta = o(\alpha)$ .

If  $\lim \frac{\beta}{\alpha} = c \neq 0$  ( $c$  is a constant), then we say that  $\beta$  and  $\alpha$  are infinitesimals of the same order.

In the particular case where  $c = 1$ , we say that  $\beta$  and  $\alpha$  are equivalent infinitesimals and denote it by  $\beta \sim \alpha$ .

When  $c = \infty$ , we say that  $\beta$  is an infinitesimal of lower order than  $\alpha$ .

For example, since  $\lim_{x \rightarrow 0} \frac{x^2}{x} = \lim_{x \rightarrow 0} x = 0$ , it follows that as  $x \rightarrow 0$ ,  $x^2$  is an infinitesimal

of higher order than  $x$ . We may denote this by  $x^2 = o(x)$ .

Conversely, since  $\lim_{x \rightarrow 0} \frac{x}{x^2} = \lim_{x \rightarrow 0} \frac{1}{x} = \infty$ , it follows that  $x$  is an infinitesimal of lower order than  $x^2$ .

For another example, since  $\lim_{x \rightarrow 0} \frac{x}{2x} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$ , it follows that as  $x \rightarrow 0$ ,  $x$  and  $2x$  are infinitesimals of the same order.

## 2.4. Two important limits

In the following, we will use two principles that limit exists to establish two important limits. With these two important limits, many problems in finding the limits of functions can be solved. And it also provides the necessary foundation for deriving the standard derivative formulae in the subsequent chapter.

### a) Two principles that limit exists

#### Principle 1

**Theorem.** If, in the change process  $x \rightarrow x_0$ , the three variables  $f(x)$ ,  $g(x)$ , and  $h(x)$  always have a relationship

$$g(x) \leq f(x) \leq h(x)$$

and  $\lim_{x \rightarrow x_0} g(x) = \lim_{x \rightarrow x_0} h(x) = A$

then  $\lim_{x \rightarrow x_0} f(x) = A$

This theorem holds generally in mathematics. For example,

$$5 \leq x \leq 5 \xleftrightarrow{\text{definite result}} x = 5$$

#### Principle 2

Consider a sequence  $u_n = f(n)$ . If, for all positive integers  $n$ , the inequality

$$f(n) < f(n+1)$$

always holds, then  $f(n)$  is a monotonically increasing sequence.

If, for all positive integers  $n$ , the inequality

$$f(n) > f(n+1)$$

always holds, then  $f(n)$  is a monotonically decreasing sequence.

If there exist two constants  $m$  and  $M$  ( $m < M$ ) such that, for all positive integers  $n$ , the inequality

$$m \leq f(n) \leq M$$

always holds, then  $f(n)$  is a bounded sequence.

**Theorem.** If a sequence  $u_n = f(n)$  is monotonic and bounded, then the limit

$$\lim_{n \rightarrow \infty} f(n) = A$$

will definitely exist, where  $A$  is a constant.

## b) Two important limits

(1) By applying principle 1, we can derive the first important limit.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

**Proof.** Consider a unit circle, as shown in Figure 2.3.

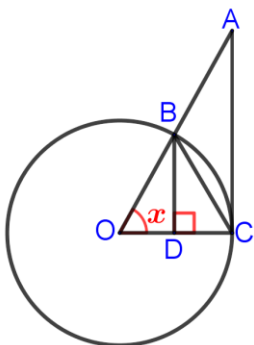


Figure 2.3

Let the central angle  $\widehat{BOC} = x$  ( $0 < x < \frac{\pi}{2}$ ).

Then area of  $\triangle OBC < \text{area of sector OBC} < \text{area of } \triangle OAC$

Since area of  $\triangle OBC = \frac{1}{2} OC \cdot BD = \frac{1}{2} \cdot 1 \cdot \sin x = \frac{1}{2} \sin x$

$$\text{area of sector OBC} = \frac{1}{2} OC \cdot OB \cdot \widehat{BOC} = \frac{1}{2} \cdot 1 \cdot 1 \cdot x = \frac{1}{2} x$$

$$\text{area of } \triangle OAC = \frac{1}{2} OC \cdot AC = \frac{1}{2} \cdot 1 \cdot \tan x = \frac{1}{2} \tan x$$

thus  $\frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x$

Dividing the inequality by  $\frac{1}{2} \sin x$ , we obtain

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

That is,  $\cos x < \frac{\sin x}{x} < 1$

Since the inequality holds when  $x$  is replaced by  $-x$ , it remains valid for all  $x$

in the interval  $0 < |x| < \frac{\pi}{2}$ .

And  $\lim_{x \rightarrow 0} \cos x = 1$ ,  $\lim_{x \rightarrow 0} 1 = 1$

By principle 1, we obtain the limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

□

**Example 1.** Find  $\lim_{x \rightarrow 0} \frac{\tan x}{x}$ .

**Solution.** 
$$\lim_{x \rightarrow 0} \frac{\tan x}{x} = \lim_{x \rightarrow 0} \frac{\sin x}{x \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \lim_{x \rightarrow 0} \frac{1}{\cos x} = 1$$

**Example 2.** Find  $\lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 6x}$ .

**Solution.** 
$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 6x} &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 6x \cos 3x} = \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 6x \cos 3x} \times \frac{18x}{18x} \\ &= \lim_{x \rightarrow 0} \frac{\sin 3x}{\sin 6x \cos 3x} \times \frac{6x}{3x} \times \frac{3}{6} = \lim_{x \rightarrow 0} \left( \frac{\sin 3x}{3x} \times \frac{6x}{\sin 6x} \times \frac{1}{\cos 3x} \times \frac{3}{6} \right) = \frac{3}{6} = \frac{1}{2} \end{aligned}$$

**Example 3.** Find  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}$ .

**Solution.** 
$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \frac{x}{2}}{2^2 \times \frac{x^2}{2^2}} = \frac{2}{4} \lim_{x \rightarrow 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2 = \frac{1}{2}$$

(2) By applying principle 2, we can derive the second important limit.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

where  $e$  is a constant. It is an irrational number, and its value is

$$e = 2.7182818284590\dots$$

**Proof.** Let  $f(n) = (1 + \frac{1}{n})^n$ .

First, we show that  $f(n)$  is monotonically increasing. Applying the binomial theorem, we obtain

$$\begin{aligned} f(n) &= (1 + \frac{1}{n})^n \\ &= 1 + 1 + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!} \cdot \frac{1}{n^3} + \dots + \frac{n(n-1)(n-2) \cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n}) + \frac{1}{3!} (1 - \frac{1}{n})(1 - \frac{2}{n}) + \dots + \frac{1}{n!} (1 - \frac{1}{n})(1 - \frac{2}{n}) \cdots (1 - \frac{n-1}{n}) \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} f(n+1) &= (1 + \frac{1}{n+1})^{n+1} \\ &= 1 + 1 + \frac{1}{2!} (1 - \frac{1}{n+1}) + \frac{1}{3!} (1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) + \dots \\ &\quad + \frac{1}{n!} (1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) \cdots (1 - \frac{n-1}{n+1}) + \frac{1}{(n+1)!} (1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) \cdots (1 - \frac{n}{n+1}) \end{aligned}$$

Comparing the two expansions, the first two terms are identical. However, from the third term onwards, each term in  $f(n+1)$  is greater than the corresponding term in  $f(n)$ . Furthermore, the expansion of  $f(n+1)$  contains one more positive term at the end. Therefore,

$$f(n) < f(n+1) \quad (n = 1, 2, 3, \dots)$$

$f(n)$  is a monotonically increasing sequence.

In the following, we show that  $f(n)$  is bounded. In the expansion of  $f(n)$ , all numbers in the brackets are less than 1. Consequently

$$f(n) < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!} + \dots + \frac{1}{n!}$$

Since  $\frac{1}{2^{k-1}} \geq \frac{1}{k!} \quad (k \geq 2)$

thus  $f(n) < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{k-2}} + \dots + \frac{1}{2^{n-1}} = 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 3 - \frac{1}{2^{n-1}} < 3$

That is,  $f(n) < 3$

It can be seen from here that no matter how big the value  $n$  is,  $f(n)$  is always less than 3. That is,  $f(n)$  is a bounded sequence. Therefore, by principle 2, we obtain the limit.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

It can be shown that for a continuous independent variable  $x$ , we also have

$$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

This formula can also be written in the form

$$\lim_{a \rightarrow 0} (1+a)^{\frac{1}{a}} = e$$

A logarithm with base  $e$  is called a natural logarithm. The natural logarithm of  $x$  is denoted by  $\ln x$ . In mathematics, we frequently employ the natural logarithmic functions,  $y = \ln x$  and exponential functions with base  $e$ ,  $y = e^x$ .

**Example 4.** Find  $\lim_{x \rightarrow \infty} \left(\frac{x+4}{x}\right)^x$ .

**Solution.**  $\lim_{x \rightarrow \infty} \left(\frac{x+4}{x}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x}{4}}\right)^{x \cdot \frac{4}{4}} = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{x}{4}}\right)^{\frac{x}{4}}\right]^4 = e^4$

**Example 5.** Find  $\lim_{x \rightarrow \infty} (1 - \frac{1}{x})^x$ .

**Solution.**  $\lim_{x \rightarrow \infty} (1 - \frac{1}{x})^x = \lim_{x \rightarrow \infty} [(1 + \frac{1}{(-x)})^{-x}]^{-1} = e^{-1}$

## Exercise 2

1. Find the following limits.

a.  $\lim_{x \rightarrow 3} (5 + x^2 - x)$

b.  $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$

c.  $\lim_{x \rightarrow 1} \frac{x - 1}{\sqrt{x + 1}}$

d.  $\lim_{x \rightarrow 1} \frac{x + 1}{x^2 - x - 2}$

e.  $\lim_{x \rightarrow 2} \frac{\sqrt{x + 7} - 3}{x - 2}$

f.  $\lim_{x \rightarrow \infty} \frac{x^3 + x^2 - 4x + 1}{2x^3 + x + 9}$

g.  $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x - 3}$

h.  $\lim_{x \rightarrow \infty} \frac{2x + 5}{x}$

i.  $\lim_{x \rightarrow 0} \frac{x}{\sqrt{x + 4} - 2}$

j.  $\lim_{x \rightarrow 0} \sin(x + \pi)$

k.  $\lim_{x \rightarrow -5} (3x + 14)^{99}$

l.  $\lim_{x \rightarrow -3} 4$

2. Find the following limits.

a.  $\lim_{x \rightarrow 1} \sqrt{8 - x}$

b.  $\lim_{x \rightarrow 2} \sqrt{x^2 + 12}$

c.  $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 2}$

d.  $\lim_{x \rightarrow 2} \frac{x^3 - 8}{x - 2}$

e.  $\lim_{x \rightarrow 9} \frac{9 - x}{3 - \sqrt{x}}$

f.  $\lim_{x \rightarrow 0} \frac{4x}{(3 + x)^2 - 9}$

g.  $\lim_{x \rightarrow -1} \frac{x^2 + 5x + 6}{x^2 + 3x + 2}$

h.  $\lim_{x \rightarrow 1} \frac{2x^2 - 10x + 8}{3x^2 - 3x}$

3. Find the following limits.

a.  $\lim_{x \rightarrow 3} \frac{x^4 - 81}{x^3 - 3x^2}$

b.  $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + 12x - 4}{x^3 - 4x}$

c.  $\lim_{x \rightarrow 2} \frac{x - 2}{\sqrt{x + 2} - 2}$

d.  $\lim_{x \rightarrow \infty} \frac{3x + 1}{3x^2 + x - 4}$

e.  $\lim_{x \rightarrow -2} \frac{x^2 + 3x + 2}{x^2 - x - 6}$

f.  $\lim_{x \rightarrow -3} \frac{\sin \frac{\pi x}{6}}{x + 1}$

g.  $\lim_{x \rightarrow 0} \frac{\sqrt{x + 4} - 2}{x}$

h.  $\lim_{x \rightarrow -8} \frac{\sqrt{1 - x} - 3}{2 + \sqrt[3]{x}}$

4. Find the following limits.

a.  $\lim_{x \rightarrow \infty} \frac{x^5 - x^4 + 3}{4x^5 - 3x^3 + 2x + 6}$

b.  $\lim_{x \rightarrow 3} \frac{x^3 - 3x^2 + x - 3}{x - 3}$

c.  $\lim_{x \rightarrow 1} \frac{x^2 - 1}{(x - 1)\sqrt{x^2 + 1}}$

d.  $\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{3 - \sqrt{2x + 1}}$

e.  $\lim_{x \rightarrow 3} \frac{2x^3 + 5x^2 - 39x + 18}{x^3 + 3x^2 - 13x - 15}$

f.  $\lim_{x \rightarrow -1} \frac{x^3 + 6x^2 + 11x + 6}{x^3 - 4x^2 - x + 4}$

5. Find the following limits.

a.  $\lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 5x}$

b.  $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^{3x}$

c.  $\lim_{x \rightarrow 0} \frac{\tan 3x}{\tan x}$

d.  $\lim_{x \rightarrow 0} \frac{x}{\sin 5x}$

e.  $\lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2}$

f.  $\lim_{x \rightarrow \infty} \left(\frac{5x + 2}{5x}\right)^x$

g.  $\lim_{x \rightarrow 0} (1 - 2x)^{\frac{1}{x}}$

h.  $\lim_{x \rightarrow 0} \frac{\tan^3 x - \sin^3 x}{x^3}$